ns251b

(2)

27/6 '08

Exam NS-251B Electrodynamics, June 2008

Duration: 3 hours

Open-book exam: no

Formula sheet: yes, 1 obtained with test, 1 prepared by you (1 A4)

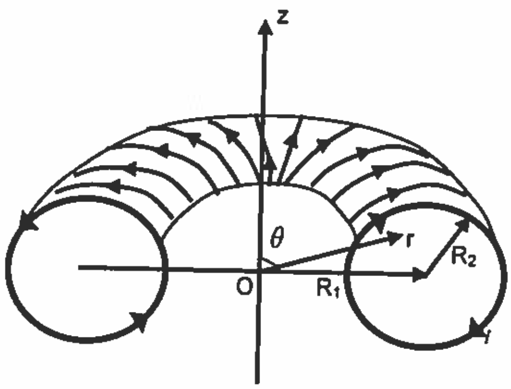
Use of calculator permitted

Grading as indicated in problems

Please give your answers to problems 1, 2, and 3 on separate sheets. Don't forget to put your name on all sheets!

Problem 1 [totaal 35 p]

A toroidal coil consists of a circular ring ("a bike tire") around which a long wire is wrapped. The radius of the ring is denoted by R_1, the number of windings by N. The windings are closely wrapped and have a circular shape with radius R_2. The coil is positioned in vacuum and carries a (free) current I. Half of the coil is shown in the figure below. We recommend using spherical coordinates in this problem.



a) [5 points] Give all symmetry properties of this current distribution. What is the consequence for the direction of the $\vec{B}$-field?

b) [7 points] Calculate the $\vec{B}$-field everywhere in space

The coil is now filled with a ferromagnetic material with relative permeability μ_r. The current I is sufficiently small to keep the properties of the material linear.

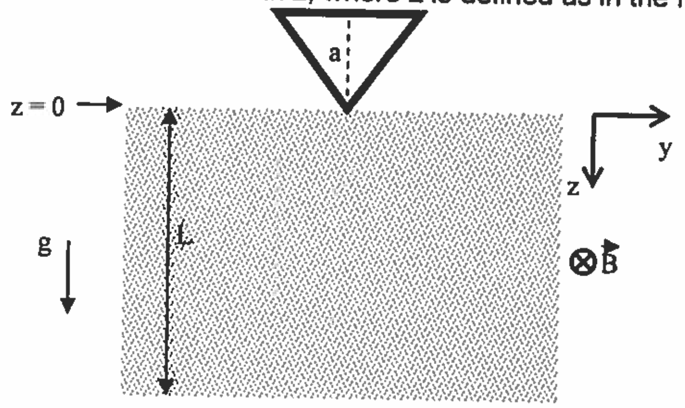
a) [8 points] Calculate first the $\vec{H}$-field everywhere in space and subsequently the $\vec{B}$-field.

b) [7 points] Calculate the magnetization $\vec{M}$ of the ferromagnetic material within the coil.

c) [8 points] Calculate the bound current densities $\vec{J}_b$ and $\vec{K}_b$. How large is the total bound current on the surface of the ferromagnetic material? Calculate the ratio of this current and the total free current (on the surface of the ferromagnetic material).

Problem 2 [totaal 25 p]

A circuit in the form of an equilateral triangle is let go at time t=0 in the gravitational field exactly above a constant homogeneous magnetic field, $\vec{B} = B\hat{x}$ (B is a positive constant). The magnetic field is perpendicular to the gravitational field with the direction into the paper. The magnetic field is present from z=0 to z=L and the position of circuit is determined by the lower corner of the triangle. The triangle has the height a (a < L), total resistance R and mass m. You may assume there is no friction and the self induction of the circuit can be neglected. The position of the circuit is indicated with z, where z is defined as in the figure.



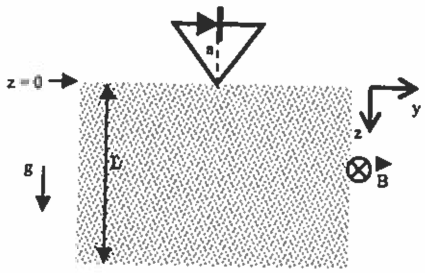
a) [7 p.] The flux change in the circuit creates an electromotive force, ε, in the circuit. Determine this EMF as a function of the velocity, v, the magnetic field $\vec{B}$, z and a in the five areas $z<0$, $0<z<a$, $a<z<L$, $L<z<(L+a)$ and $z>(L+a)$

b) [3 p.] Determine the current I in the circuit as a function of the same parameters in the areas. Don't forget to indicate the direction of the current.

c) [5 p.] Determine the total force exerted on the circuit as a function of the same parameters.

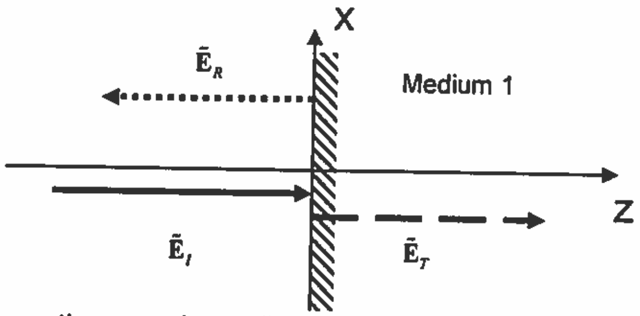
d) [5 p.] Which one of the answers in a)-c) change if we reverse the magnetic field ($\vec{B} = -B\hat{x}$)?

e) [5 p.] We now place a diode in the upper leg of the triangle (current can only flow in the direction of the arrow, see figure. Give the answer to a) and b) in this situation. Does the direction of the $\vec{B}$ field have any influence on the answer?



Opgave 3 [totaal 40 p]

A mono chromomatic plane wave in vacuum ($\tilde{\mathbf{E}}_I(z,t) = \tilde{\mathbf{E}}_{0I} e^{i(k_I \cdot z - \omega t)}$; $\tilde{\mathbf{B}}_I(\mathbf{r},t) = \frac{1}{c}(\hat{\mathbf{k}}_I \times \tilde{\mathbf{E}}_I)$) approaches the interface of medium 1 perpendicularly (z=0 – see the drawing). The medium is a linear homogenous Ohmic conductor ($\vec{J}_f = \sigma \vec{E}$). We denote the permittivity, permeability, velocity of light and refractive index of medium 1 with, $\varepsilon_1, \mu_1 (= \mu_0), \mathrm{v}_1, \mathrm{n}_1$. $\varepsilon_0 = 8{,}85 \cdot 10^{12}\,\mathrm{C^2/Nm^2}$ and $\mu_0 = 4\pi \cdot 10^{-7}\,\mathrm{N/A^2}$.



The Maxwell equations can in medium 1 be written as:

$$(i) \quad \nabla \cdot \mathbf{E} = \rho_f / \varepsilon_1 \qquad (ii) \quad \nabla \cdot \mathbf{B} = 0$$

$$(iii) \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (iv) \quad \nabla \times \mathbf{B} = \mu_0 \sigma \mathbf{E} + \mu_0 \varepsilon_1 \frac{\partial \mathbf{E}}{\partial t}$$

a) [2 p.] Explain why ρ_f normally can be neglected.

b) [5 p.] Determine the wave equations for the **E** en **B** fields in medium 1 (hint: determine the rotation of *(iii)* and *(iv)*). Show further that the plane waves

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{-\kappa z} e^{i(kz - \omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{-\kappa z} e^{i(kz - \omega t)}$$

with

$$k \equiv \omega \sqrt{\frac{\varepsilon \mu_0}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\varepsilon \omega} \right)^2} + 1 \right]^{1/2}, \quad \kappa \equiv \omega \sqrt{\frac{\varepsilon \mu_0}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\varepsilon \omega} \right)^2} - 1 \right]^{1/2}$$

Are solutions to the wave equation for the **E** and **B** fields.

c) [5 p.] Determine the skin depth, $d \equiv 1/\kappa$, for a good conductor $\left(\frac{\sigma}{\varepsilon\omega}\right) \gg 1$ and for a poor conductor $\left(\frac{\sigma}{\varepsilon\omega}\right) \ll 1$.

d) [5 p.] Determine the reflected wave (hint: don't forget the phase jump of π for $n_1 > n$).

e) [5 p.] Determine the position of the maxima and minima for $\langle E^2 \rangle$ as a function of the distance to the interface for z<0.

f) [5 p.] Determine the reflection coefficient R $(= I_R / I_I)$

g) [5 p.] Determine the transmitted fields (don't forget the phase difference between $\tilde{\mathbf{E}}_T$ and $\tilde{\mathbf{B}}_T$).

h) [5 p.] Determine the Poynting vector as a function of z for z>0.

i) [3 p.] A microwave oven uses a frequency of 2.45 GHz. A roast beef has, at this frequency, a value of $\varepsilon/\varepsilon_0 \approx 50$ and $\sigma = 2\,\Omega^{-1}\text{m}^{-1}$. Determine the skin depth and the transmission T. An infrared oven has a wave length of 1 mm and $\sigma = 2\,\Omega^{-1}\text{m}^{-1}$ also at this wavelength; does it matter whether you use an infrared oven or a microwave oven? How would you suggest the "chef" in the family to prepare a roast beef?

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence: $$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl: $$\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla\times\mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint\mathbf{A}\cdot d\mathbf{a}$

Curl Theorem: $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint\mathbf{A}\cdot d\mathbf{l}$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85\times10^{-12}\ \mathrm{C^2/Nm^2}$		(permittivity of free space)
$\mu_0 = 4\pi\times10^{-7}\ \mathrm{N/A^2}$		(permeability of free space)
$c = 3.00\times10^{8}\ \mathrm{m/s}$		(speed of light)
$e = 1.60\times10^{-19}\ \mathrm{C}$		(charge of the electron)
$m = 9.11\times10^{-31}\ \mathrm{kg}$		(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2+y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$