Instituut voor Theoretische Fysica, Universiteit Utrecht

MID-EXAM ADVANCED QUANTUM MECHANICS

November 10, 2011

- The duration of the exam is 3 hours.
- The exam is closed-book.
- Usage of a calculator and a dictionary is allowed.
- Use different sheets for each exercise.
- Write your name and initials on every sheet handed in.
- Divide your available time wisely over the exercises.

Problem 1 (15 points)

Consider a pure state described by the following wave function

$$\psi(x) = Ce^{\frac{ip_0 x}{\hbar} - \frac{(x-x_0)^2}{2a^2}},$$

where p_0, x_0 and a are real parameters. Determine the average values of position and momentum as well as their variance

$$(\sigma_{Q,\psi})^2 = \langle\psi|Q^2|\psi\rangle - \langle\psi|Q|\psi\rangle^2, \qquad (\sigma_{P,\psi})^2 = \langle\psi|P^2|\psi\rangle - \langle\psi|P|\psi\rangle^2.$$

Is the uncertainty principle satisfied?

Problem 2 (25 points)

Let $\mathcal{H} = L^2([0,1])$ and consider the operators T_0 and $T_{0,0}$ defined on the following domains

$$\begin{aligned} D(T_0) &= \{\psi \in \mathcal{H} : \quad \psi \text{ suitably smooth}, \ \psi(0) = \psi(1) = 0\}, \\ D(T_{0,0}) &= \{\psi \in \mathcal{H} : \quad \psi \text{ suitably smooth}, \ \psi(0) = \psi(1) = 0 = \psi'(0) = \psi'(1)\}, \end{aligned}$$

(where the prime indicates the derivative) and acting as

$$\begin{aligned} T_0\psi(x) &= -\frac{d^2}{dx^2}\psi(x) = -\psi''(x), \quad \forall\ \psi \in D(T_0), \\ T_{0,0}\psi(x) &= -\frac{d^2}{dx^2}\psi(x) = -\psi''(x), \quad \forall\ \psi \in D(T_{0,0}). \end{aligned}$$

1. Find the adjoint of the operator T_0, and the domain $D(T_0^\dagger)$ (do not consider smoothness issues).

2. Find the adjoint of the operator $T_{0,0}$, and the domain $D(T_{0,0}^\dagger)$ (do not consider smoothness issues).

3. State which one of these two operators is self-adjoint, and, for that operator, find the spectrum, i.e. an explicit formula for the eigenstates $\psi_n(x)$ (up to a non-zero normalization constant) and the corresponding eigenvalues λ_n, where n is an appropriate index.

4. Show that $(\psi_n, \psi_m) = \delta_{mn}$, up to a normalization constant, where the brackets denote the scalar product; the convention for (anti-)linearity used in the formula is inessential here.

   Hint: it is possible (but not mandatory) to answer to this point without evaluating any integral.

<u>Problem 3</u> (25 points)

Consider the following Weyl operators

$$\begin{aligned} U(u) &= e^{-iuP}, \\ V(v) &= e^{-ivQ}, \end{aligned}$$

where u and v are two real numbers (parameters) and (P, Q) are the operators of momentum and coordinate satisfying the Heisenberg commutation relations. The Weyl quantization map associates to a real function $f \equiv f(p,q)$ the following self-adjoint operator

$$A_f = \frac{1}{2\pi} \int_{\mathbf{R}^2} \mathrm{d}u \mathrm{d}v \, \hat{f}(u,v) \, e^{\frac{ihuv}{2}} \, V(v) U(u) \, ,$$

where $\hat{f}(u,v)$ is the Fourier image of $f(p,q)$.

1. Find the action of $U(u)$ and $V(v)$ on a wave function in the coordinate representation.

2. Find the kernel of the operator A_f in the coordinate representation.

3. Express the trace of the operator A_f in terms of $f(p,q)$.

Problem 4 (35 points)

Consider the one-dimensional *classical* harmonic oscillator. Let m be the mass, and ω the angular frequency.

1. Consider the complex function of momentum and position
$$\alpha(t) = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{m\omega}\, q(t) + \frac{i}{\sqrt{m\omega}}\, p(t) \right) .$$
Find the evolution equation it satisfies, that is, compute $\frac{d}{dt}\alpha(t)$. Solve the resulting differential equation with the initial condition $\alpha(0) = \alpha$, where $\alpha \in \mathbb{C}$.

Consider now the one-dimensional *quantum* harmonic oscillator.

2. Consider the raising and lowering operators $a^\dagger$ and a, given by
$$a^\dagger = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{m\omega}\, Q - \frac{i}{\sqrt{m\omega}}\, P \right), \qquad a = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{m\omega} Q + \frac{i}{\sqrt{m\omega}} P \right).$$
State what the commutators $[a^\dagger, a]$ and $[a, a^\dagger]$ are. Write the Hamiltonian operator H in terms of $a, a^\dagger$ and give the spectrum $\{\lambda_n\}_{n\in\mathbb{N}}$ of H. Show that the normalized eigenvectors are
$$|\psi_n\rangle = \frac{1}{\sqrt{n}} a^\dagger |\psi_{n-1}\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |\psi_0\rangle, \qquad H|\psi_n\rangle = \lambda_n |\psi_n\rangle.$$
You may assume that $|\psi_0\rangle$ is the unique normalized vector such that $a|\psi_0\rangle = 0$.

3. Consider the following eigenvalue equation
$$a\,|\phi_\alpha\rangle = \alpha\,|\phi_\alpha\rangle, \quad \alpha \in \mathbb{C}\,.$$
Using the properties of $a^\dagger, a$ and $|\psi_n\rangle$, show that
$$|\phi_\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\, |\psi_n\rangle\,.$$
Show also that $\langle \phi_\alpha | \phi_\alpha \rangle = 1$. State whether it must be $\alpha \in \mathbb{R}$ and if yes, why.

4. Find $|\phi_\alpha(t)\rangle$, the time evolution of $|\phi_\alpha\rangle$ in the Schroedinger picture. Show that up to a global phase this is a still an eigenfunction of a, with eigenvalue $\alpha(t)$:
$$|\phi_\alpha(t)\rangle = e^{i\Phi(t)} |\phi_{\alpha(t)}\rangle\,.$$
Determine $\Phi(t)$ and $\alpha(t)$. Which one of these functions is physically relevant?

5. Write Heisenberg's uncertainty principle for the rescaled position and momentum operators $\tilde{Q} = \sqrt{m\omega}Q$ and $\tilde{P} = \frac{1}{\sqrt{m\omega}}P$ in general. Compute what their uncertainty on the state $|\phi_\alpha\rangle$ is, that is, compute

$$\sigma_{\tilde{P},\phi_\alpha}\,\sigma_{\tilde{Q},\phi_\alpha}\,,$$

where as usual $\sigma^2_{A,\psi} = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2$. Explain how what you found for $|\phi_\alpha\rangle$ can be extended to $|\phi_\alpha(t)\rangle$.

6. For a fixed value of α, consider the operator

$$U_\alpha = \exp\left[\alpha\, a^\dagger - \alpha^* a\right]\,,$$

and show that it is unitary. Express it in terms of a multiple of the operator $B_\alpha A_\alpha$, where

$$A_\alpha = \exp\left[-\alpha^* a\right] \quad \text{and} \quad B_\alpha = \exp\left[\alpha\, a^\dagger\right]\,,$$

and show that

$$|\phi_\alpha\rangle = U_\alpha|\psi_0\rangle\,.$$

Explain how this this fact yields an independent check that $|\phi_\alpha\rangle$ is normalized.