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IN ELEKTRONISCHE VORM BESCHIKBAAR GEMAAKT DOOR DE $\mathcal{TBC}$ VAN A–ESKWADRAAT.
HET COLLEGE NS-364B WERD IN 2008/2009 GEGEVEN DOOR DR. GLEB ARUTYONOV.

Classical field theory (NS-364B)
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Opgave 1

Consider an interaction scalar field $\phi(x)$ on the 3+1-dimensional Minkowski space-time described by the following action

$$S = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \lambda \phi^4 \right)$$

where λ is a constant (called the 'coupling constant').

a) Show that the action is invariant under infinitezimal transformations

$$\phi \to \phi + \delta\phi, \quad \delta\phi = \varepsilon(x^\mu \partial_\mu \phi + \phi),$$

up to a total derivative term $\delta S = \varepsilon \int d^4x \partial_\mu F^\mu$, where ε is a constant small (infinitezimal) parameter. Find the vector F^μ explicitly.

b) Construct the corresponding Noether current J^μ by using the general expression from the lecture notes. You can check, however, that this current is nog conserved, i.e. $\partial_\mu J^\mu \neq 0$. The reason for this non-conservation is that the action S is not *exactly* invariant unter infinitezimal transformations of ϕ, but it is invariant up to a total derivative term.

c) Show that an improved current

$$J^\mu_{improved} = J^\mu - F^\mu$$

is conserved due to equations of motion.

Opgave 2

A scalar $\phi(x)$ and a vector $A^i(x)$ are the quantities which under general transformations of coordinates $x^i \to x'^i(x^j)$ transform as follows

$$\phi(x) \to \phi'(x') = \phi(x), \quad A^i(x) \to A'^i(x') = \frac{\partial x'^i}{\partial x^j} A^j(x)$$

A *pseudoscalar* and *pseudovector* are the quantities which transform in a diffent way

$$\begin{aligned} \phi(x) &\to \phi'(x') = det\left(\frac{\partial x'^i}{\partial x^j}\right)\phi(x), \\ A^i(x) &\to A'^i(x') = det\left(\frac{\partial x'^i}{\partial x^j}\right)\frac{\partial x'^i}{\partial x^j} A^j(x). \end{aligned}$$

In particular, under space reflection $\vec{x} \to -\vec{(x)}$ a scalar and a vector transform as

$$\phi(x) \to \phi'(t, -\vec{x}) = \phi(t, \vec{x}), \quad A^i(x) \to A'^i(t, -\vec{x}) = -A'^i(t, \vec{x}),$$

while a pseudovector and a pseudoscalar transform as

$$\phi(x) \to \phi'(t, -\vec{x}) = -\phi(t, \vec{x}), \quad A^i(x) \to A'^i(t, -\vec{x}) = A'^i(t, \vec{x}).$$

As an example, in three dimensions, if $\vec{A}$ and $\vec{B}$ are vectors, then $\vec{A} \times \vec{B}$ is a pseudovector and $\left(\vec{A} \cdot \vec{B}\right)$ is a scalar. Also, if $\vec{C}$ is a pseudovector, then $\vec{A} \times \vec{C}$ is a vector and $\left(\vec{A} \cdot \vec{C}\right)$ is a pseudoscalar.

a) Let $\vec{A}$ be a vector and $\vec{B}$ be a pseudovector. Derive wether the following quantities are vectors, pseudovectors, scalars or pseudoscalars
$rot\vec{A}$, $rot\vec{B}$, $div\vec{A}$, $div\vec{B}$

b) Using the second pair of Maxwell's equations and the definitions of the charge density ρ and the current density $\vec{j}$, determine whether ρ, $\vec{j}$, $\vec{E}$ and $\vec{H}$ are scalars, pseudoscalars, vectors or pseudovectors.

Opgave 3

Consider the action for electromagnetic field A_μ:

$$S = -\tfrac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}$$

Which symmetries of this action you know? Use this action to obtain the equations of motion for A_μ.

Opgave 4

Consider the following vector and scalar potentials

$$\vec{A}(x,t) = \vec{A}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}, \quad \varphi(x,t) = 0,$$

where $\vec{A_0}$ and $\vec{k}$ are constant three-dimensional vectors, and ω is a constant frequency.

a) Derive the electric and magnetic fields corresponding to these potentials

b) Determine the conditions imposed on $\vec{A_0}$, $\vec{k}$ and ω by Maxwell's equations assuming that the absence of charge and current densities, i.e. $\rho = 0$ and $\vec{j} = 0$.

Opgave 5 (Bonus problem!)

Consider a charge density $\rho(x,t)$ and a current density $\vec{j}(x,t)$ in vacuum. Show that in the Coulomb gauge $div\vec{A} = 0$, the vector potential is determined by the transverse part of the current $\vec{j}^{\perp}$ only.
Hint 1 The current is decomposed on the transversal $\vec{j}^{\perp}$ and the longitudinal $\vec{j}^{\parallel}$ parts

$$\vec{j} = \vec{j}^{\perp} + \vec{j}^{\parallel}$$

where $div\vec{j}^{\perp} = 0$ and the longitudinal part fulfills $rot\vec{j}^{\parallel} = 0$.
Hint 2 Use the continuity equation to express the scalar potential.