Midterm Exam - "Quantum Matter" (NS-371B)

April 16, 2012
Duration of the exam: 3 hours

1. Use a separate sheet for every exercise.
2. Write your name and initials in all sheets, on the first sheet also your address and your student ID number.
3. Write clearly, unreadable work cannot be corrected.
4. You are NOT allowed to use any kind of books or lecture notes. A list with some useful formulas is given at the end of the exam sheet.

1. Magnons in Antiferromagnets

Ferromagnetic and antiferromagnetic materials have excitations called "spin waves" or "magnons", corresponding to oscillations in the magnetization direction that (classically) behave as waves. As you have seen in the exercises, in 1D *ferromagnets*, the dispersion relation is

$$\varepsilon_{\mathbf{k}} = J\left[1 - \cos(k_x a)\right] \ , \tag{1}$$

where $J > 0$ is the so-called "spin stiffness" and a is the lattice constant. On the other hand, in 1D *antiferromagnets* the dispersion relation is

$$\varepsilon_{\mathbf{k}}^2 = (-J)^2\left[1 - \cos^2(k_x a)\right] \tag{2}$$

and now $J < 0$. The aim here is to understand how the heat capacity of *antiferromagnets* depends on the temperature.

1. (1.0) Given the fact that magnons obey the Planck distribution function, determine their contribution to the heat capacity at low temperatures in one dimension. *Hint:* expand the dispersion for small k_x, determine the internal energy U, and then determine the heat capacity C_v. Write the integral in terms of dimensionless variables and call it I_{1D}.

2. (1.0) Same as (a), but now in two and three dimensions. Call the integrals I_{2D} and I_{3D}. Guess the expression for ε_k in two and three dimensions, looking at Eq.(2).

3. (1.0) How is the temperature dependence of C_v for spin-waves in d-dimensions? Compare the results for antiferromagnets with the results you know for phonons.

2. 2D Bose-Einstein Condensation

In this exercise we will consider a two dimensional (2D) Bose gas with spin zero.

1. (1.0) Show that the surface density $n = N/A$ of a homogeneous 2D Bose gas is given by,
$$\Lambda_{th}^2 n = -\ln\left[1 - \exp(\mu/k_B T)\right], \quad (3)$$
where $\Lambda_{th} = (h^2/2\pi m k_B T)^{1/2}$ is the thermal de Broglie wavelength.

2. (1.0) Argue, using (3), that a homogeneous 2D Bose gas does not condense to a BEC.

3. (1.0) We can bring a 2D Bose gas in thermal contact with a classical three dimensional (3D) ideal gas. For example, the 2D gas can be adsorbed on a film of superfluid ^{3}He, and then it becomes part of a 3D volume containing a classical ideal gas. Calculate the chemical potential of this 3D gas as a function of its density and temperature. *Hint:* Calculate the number of particles $N(V, T, \mu)$ and invert this expression to obtain μ. Try to get a Gaussian integral.

3. Heat capacity of liquid ^{4}He

Liquid ^{4}He, which are bosonic atoms, becomes superfluid below a temperature of 2.17 K. Just as in a crystal, the low energy excitations of liquid ^{4}He are sound waves, whose quanta are the phonons. However, in liquid ^{4}He there is also another type of elementary excitation called the roton, see Figure 1. The dispersion relations for the phonons and rotons are, respectively,

$$\begin{aligned} \epsilon_p &= p c_1, \\ \epsilon_r &= \Delta + \frac{(p - p_0)^2}{2\mu_r}, \end{aligned}$$

where c_1 is the sound velocity, Δ is the energy gap of the rotons, μ_r is the effective mass of the rotons, and $p = \hbar k$ is the momentum of the atoms. As we have seen in crystals, the low energy excitations determine the behaviour of the thermodynamic quantities, such as the specific heat. In this sense liquid ^{4}He can be described as an ideal gas of phonons and an ideal gas of rotons.

1. (1.0) Let us first concentrate on the phonon gas. Calculate the temperature dependence of the free energy of the phonon gas, which is given by the free energy of the ideal Bose gas
$$F = -k_B T \int \ln(1 + n) \frac{d^3 p}{(2\pi\hbar)^3},$$

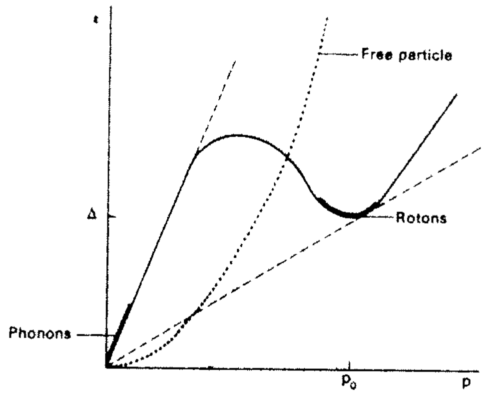

Figure 1: Spectrum of ^{4}He, with energy (vertical axis) versus momentum (horizontal axis), showing the phonon and roton excitations.

where $n = 1/(e^{\beta\epsilon} - 1)$ is the Planck distribution. Hint: write it as a dimensionless integral and extract the temperature dependence without solving the integral explicitly. Recall that d^3p is an infinitesimal volume element of a sphere.

2. (1.0) Calculate the temperature dependence of the entropy and specific heat of the phonon gas.

3. (1.0) Consider now the roton gas. Argue that if $\Delta \gg k_B T$ the Maxwell-Boltzmann distribution can be used for the rotons instead of the Bose-Einstein distribution function. Furthermore, show that the free energy of the ideal Bose gas reduces to the free energy of a Boltzmann gas

$$F \approx -k_B T \int n_B \frac{d^3p}{(2\pi\hbar)^3},$$

where n_B is the Boltzmann distribution.

4. (1.0) Calculate the temperature dependence of the free energy of the roton gas by approximating the integral assuming that $p_0 \gg \sqrt{\frac{\mu_r}{\beta}}$. Notice that the main contribution from the Gaussian term comes from $p \approx p_0$. Show that

$$F \approx -\frac{4\pi k_B T}{(2\pi\hbar)^3} e^{-\Delta/k_B T} p_0^2 \sqrt{2\pi\mu_r k_B T}.$$

As before, you could then determine the entropy and the specific heat, but you don't need to do it here.

Formulas

- Maxwell-Boltzmann distribution: $g(\varepsilon) \propto \exp(-\varepsilon/k_B T)$
- Gibbs distribution:
$$P_i = \frac{\exp\{\beta(\mu N_i - E_i)\}}{\mathcal{Z}}, \qquad \mathcal{Z} = \sum_i \exp\{\beta(\mu N_i - E_i)\}$$
- For the different ensembles: $\Omega = e^{\beta TS}$, $Z = e^{-\beta F}$, $\mathcal{Z} = e^{-\beta \Phi_G}$, where S is the entropy, T is the temperature, $\beta = (k_B T)^{-1}$, $F = -k_B T \ln Z$ is the Helmholtz function, and $\Phi_G = -k_B T \ln \mathcal{Z} = F - \mu N = -pV$ is the grand-potential.
- Recall the usual relations from quantum mechanics:
$$E = \hbar\omega, \qquad p = \hbar k, \qquad E = p^2/2m$$
- Photons:
$$\omega = ck, \qquad k = 2\pi/\lambda, \qquad c = \lambda\nu$$
$$E = \hbar kc = pc$$
- Planck distribution:
$$f(E) = \frac{1}{e^{\beta E} - 1}$$
- Fermi-Dirac and Bose-Einstein distributions:
$$\ln \mathcal{Z} = \pm \sum_i \ln\left(1 \pm e^{\beta(\mu - E_i)}\right)$$
$$f(E) = \frac{1}{e^{\beta(E-\mu)} \pm 1},$$
where the sign $+$ stands for fermions and the sign $-$ stands for bosons.
- Polylogarithm function $\mathrm{Li}_n(z)$:
$$\int_0^\infty dx \frac{x^{n-1}}{z^{-1}e^x \pm 1} = \mp\Gamma(n)\mathrm{Li}_n(\mp z), \qquad \mathrm{Li}_n(1) = \zeta(n),$$
where ζ is the Riemann zeta-function, and $\Gamma(n) = (n-1)!$ for n a positive integer.
$$\mathrm{Li}_1(z) = \ln\left(\frac{1}{1-z}\right)$$
- Gaussian integral:
$$\int_{-\infty}^{+\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$