Trial Final Exam- "Quantum Matter" (NS-371B)

June 18 2012

Duration of the exam: 3 hours

1. Use a separate sheet for every exercise.

2. Write your name and initials in all sheets, on the first sheet also your address and your student ID number.

3. Write clearly, unreadable work cannot be corrected.

4. You are NOT allowed to use any kind of books or lecture notes. A list with some useful formulas is given at the end of the exam sheet.

1 ^{3}He and ^{4}He Mixtures

1. The atom ^{3}He has spin 1/2 and is a fermion. ^{3}He liquid can be considered as a gas of non-interacting fermions.

   (a) **(0.5)** Show that when a single orbital is filled,

   $$\left\langle (\Delta N)^2 \right\rangle = \langle N \rangle \left(1 - \langle N \rangle\right),$$

   if $\langle N \rangle$ is the average number of fermions in that orbital. By definition, $\Delta N \equiv N - \langle N \rangle$. What happens with the fluctuation for orbitals with energies deep enough below the Fermi energy so that $\langle N \rangle = 1$? For what values of $\langle N \rangle$ do the fluctuations vanish? For what value of $\langle N \rangle$ are they maximal? Can you understand these results from a physical point of view? Explain!

2. ^{4}He atoms are bosons.

   (a) **(1.0)** Using that the average number of bosons $< N >$ in diffusive contact with a reservoir is $\langle N \rangle = \frac{k_B T}{Z_{gr}} \left(\frac{\partial Z_{gr}}{\partial \mu} \right)_{T,V}$, show that

   $$\left\langle (\Delta N)^2 \right\rangle = \langle N \rangle \left(1 + \langle N \rangle\right)$$

   Hint: Prove that

   $$\left\langle (\Delta N)^2 \right\rangle = \frac{1}{\beta} \frac{\partial}{\partial \mu} \langle N \rangle$$

   (b) **(0.5)** What happens when $\langle N \rangle \gg 1$? Compare with the results for ^{3}He .

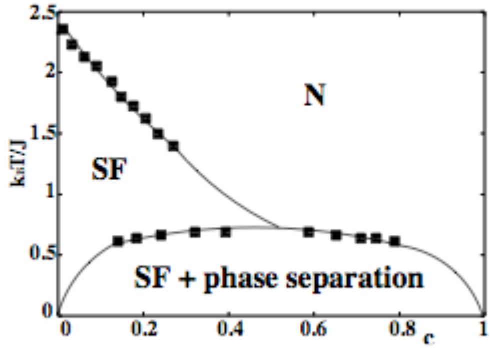

Figure 1: Phase diagram of the Blume-Emergy-Griffths model, obtained by Monte Carlo simulations. N denotes the normal phase, SF the superfluid phase.

3. Since ^{4}He atoms are bosons, they undergo a phase transition into a superfluid state below the critical temperature T_c. However, when ^{4}He atoms are mixed with ^{3}He ones, a more complex phase diagram emerges, see Fig. 1.

   The next answers must be given in terms of the Landau free energy for describing phase transitions.

   (a) **(0.5)** Describe the second order phase transition from the normal to the superfluid phase and discuss what happens with the coefficients of the free energy expansion. Determine the order parameter and discuss which derivative of the Gibbs free energy is discontinuous.

   (b) **(0.5)** Discuss the behavior of the free energy for a first order phase transition. What is the name of the point at which the second order phase transition line meets the first order line?

2 Infinite Range Ising Model

We consider an Ising model in which *all* spins $\sigma_i = \pm 1$ interact with all other spins with interaction strength $-J/N$, where N is the number of spins in the system. In the presence of a magnetic field H, the partition function is given by

$$Z(\beta, H, N) = \sum_{\{\sigma_i\}} \exp\left\{ \frac{\beta J}{2N} \sum_{i,j} \sigma_i \sigma_j + \beta H \sum_i \sigma_i \right\}, \tag{1}$$

where $\sum_{\{\sigma_i\}}$ denotes the sum over all of the 2^N spin configurations of the system.

(a) **(0.5)** Show that

$$\sum_{\{\sigma_i\}} e^{\beta(J\mu+H)\sum_i \sigma_i} = 2^N \prod_{i=1}^{N} \{\cosh\left[\beta(J\mu+H)\right]\}, \tag{2}$$

where μ is some auxiliary field which we need later.

(b) **(0.5)** Using the Gaussian identity,

$$\exp\left\{\frac{\beta J}{2N}\left(\sum_i \sigma_i\right)^2\right\} = \int_{-\infty}^{\infty} \frac{d\mu}{\sqrt{\frac{2\pi}{N\beta J}}} \exp\left\{-\frac{N\beta J}{2}\mu^2 + \beta J\mu \sum_{i=1}^{N} \sigma_i\right\}, \tag{3}$$

and equation (2), show that equation (1) can be written as

$$Z(\beta, H, N) = \int_{-\infty}^{\infty} \frac{d\mu}{\sqrt{\frac{2\pi}{N\beta J}}} \exp\left\{-\frac{N\beta J}{2}\mu^2 + N\log\left[2\cosh\left(\beta(H+\mu J)\right)\right]\right\}. \tag{4}$$

The magnetization m and susceptibility χ can be found from equation (4) by taking derivatives with respect to the magnetic field H.

(c) **(0.5)** The average magnetization is given as

$$\langle m\rangle = \frac{1}{N\beta}\frac{\partial \log Z}{\partial H} = \langle f(\mu, H, \beta)\rangle, \tag{5}$$

where the average is defined as

$$\langle \ldots \rangle = \frac{1}{Z}\int_{-\infty}^{\infty} \frac{d\mu}{\sqrt{\frac{2\pi}{N\beta J}}} \ldots \exp\left\{-\frac{N\beta J}{2}\mu^2 + N\log\left[2\cosh\left(\beta(J\mu+H)\right)\right]\right\}. \tag{6}$$

Calculate the function $f(\mu, H, \beta)$.

(d) **(0.5)** Calculate the susceptibility in the limit $N \to \infty$. The susceptibility is defined to be

$$\chi = \frac{1}{N}\frac{\partial\langle m\rangle}{\partial H}.$$

Show that it is given by

$$\chi = \beta\left[\left\langle \tanh^2(\beta(J\mu+H))\right\rangle - \langle \tanh(\beta(J\mu+H))\rangle^2\right].$$

Hint: the limit $N \to \infty$ should be taken in the last step.

(e) **(1.0)** Could you have expected the form you found for the magnetic susceptibility?

It is useful to define the function $\mathcal{F}(\mu, H)$,

$$\mathcal{F}(\mu, H) \equiv \frac{J}{2}\mu^2 - \frac{1}{\beta}\log\left[2\cosh(\beta(J\mu + H))\right], \tag{7}$$

such that

$$Z(\beta, H, N) = \int_{-\infty}^{\infty} \frac{d\mu}{\sqrt{\frac{2\pi}{N\beta J}}} e^{-\beta N \mathcal{F}(\mu, H)}. \tag{8}$$

In the thermodynamic limit, $N \to \infty$, the stationary phase method applied to Z becomes exact (for this model). The condition for stationarity is

$$\frac{\partial}{\partial \mu}\mathcal{F}(\mu, H) = 0. \tag{9}$$

The minima μ_i, where $i \in \{1, \ldots, n\}$ and n is the number of minima, found by the above condition give the dominant contributions to the partition function. Since for this model system the stationary phase method is exact in the thermodynamic limit $N \to \infty$, we can write the partition function as

$$Z(\beta, H, N) = \sum_{i=1}^{n} e^{-\beta N \mathcal{F}(\mu_i, H)}.$$

Note that we do not have to perform the integration over μ anymore! Moreover, if there is *one* absolute minimum μ_0, then in the thermodynamic limit the average magnetization will correspond to $m = \mu_0$. This means that the original auxiliary field μ, is actually the magnetization.

(f) **(0.5)** Show that the stationarity condition in equation (9) gives

$$\mu = \tanh\left[\beta(J\mu + H)\right] \tag{10}$$

(g) **(0.8)** Figure 2 shows the Landau free energy versus μ. What is the order of this phase transition? Justify your answer.

(h) **(0.7)** We now want to calculate the susceptibility at zero field, $H = 0$, using the stationary phase method. Compute the susceptibility by differentiating equation (10) w.r.t. H, and show that it yields

$$\chi_0 = \frac{1}{\beta}\frac{\partial \mu}{\partial H}\bigg|_{H=0} = \frac{1 - m^2}{1 - \beta J(1 - m^2)},$$

where the magnetization satisfies $m = \tanh(\beta J m)$. *Hint:* On both sides of equation (10) we have that the auxiliary field depends on the magnetic field, $\mu = \mu(H)$.

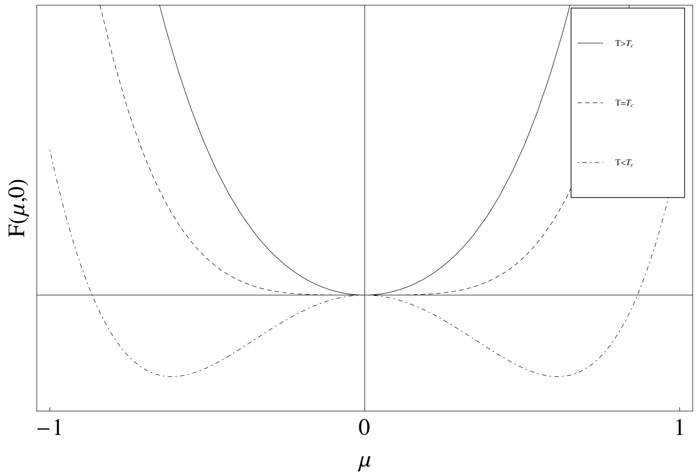

Figure 2: The Landau free energy versus μ for $H = 0$.

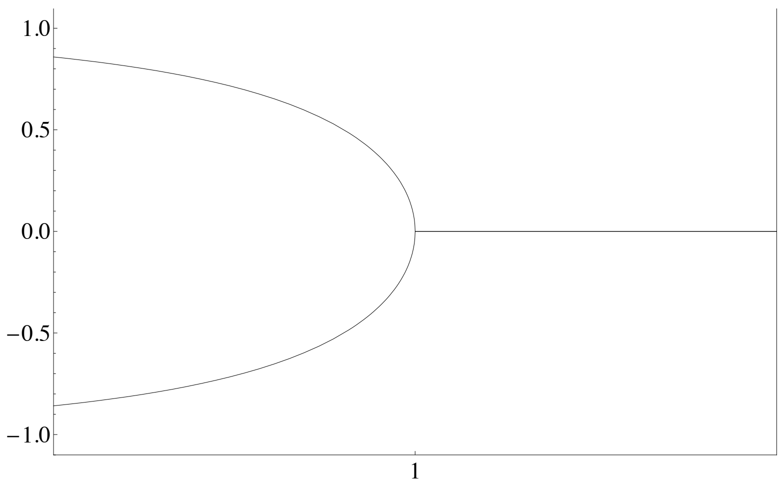

Figure 3: Something versus something...

For $T > T_c$ there is only one minimum μ_0 which corresponds to zero magnetization $m = 0$. This state is called the paramagnetic phase. This means that the susceptibility is given by

$$\chi_0^{para} = \frac{k_B T}{k_B T - J},$$

(i) **(1.0)** Calculate the critical temperature for this phase transition at $H = 0$. Could you have anticipated the behavior of the susceptibility at T_c by looking at figure 2?

(j) **(0.5)** Knowing that figure 3 is in relation with figure 2, identify what is being plotted in the horizontal and vertical axis in figure 3. Explain your answer.

(k) **(0.5)** Close to the critical temperature T_c, we can expand $\tanh(x) \approx x - x^3/3$. Give the value of the critical exponent ν, which is defined to be

$$\mu \overset{T \to T_c}{\sim} \left(\frac{T_c - T}{T}\right)^{\nu}.$$

End

Formulas

- Maxwell-Boltzmann distribution: $g(\varepsilon) \propto \exp(-\varepsilon/k_B T)$
- Planck distribution: $f(E) = \frac{1}{e^{\beta E} - 1}$
- Fermi-Dirac- en Bose-Einstein distribution: $f(E) = \frac{1}{e^{\beta(E-\mu)} \pm 1}$, where the sign $+$ stands for fermions and $-$ for bosons.
- Gaussian integral: $\int_{-\infty}^{+\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}$
- Stirling approximation: $\log(n!) \approx n \log n - n$