Departement Natuur- en Sterrenkunde
Universiteit Utrecht

MID-TERM EXAM Quantum Field Theory - NS-TP401M

Thursday, November 10, 2016, 13:30-16:30, Educatorium, Megaron.

1) Start every exercise on a separate sheet. Write on each sheet: your name and initials, and your studentnumber.
2) Please write legibly and clear. Unreadable handwriting cannot be marked!
3) The exam consists of **three** exercises and counts for 30% of the total final mark.
4) No lectures notes or any other material (books, calculators, ...) are allowed. A formularium instead is given.

Formularium

The conjugate momentum of a degree of freedom q for a Lagrangian $L(q, \dot{q})$ is given by $p = \partial L/\partial \dot{q}$. The Hamiltonian is then defined by $H = p\dot{q} - L$. Upon quantization, we impose the commutation relation $[q, p] = i\hbar$. When using natural units, we can set $c = \hbar = 1$. The time dependence of an operator O in the Heisenberg picture can be found from the Heisenberg equation $dO/dt = i[H, O]$.

The Dirac delta function and step functions have integral representations given by $(\epsilon \to 0^+)$

$$\int_{-\infty}^{+\infty} \mathrm{d}p\, e^{ipx} = 2\pi\delta(x) , \qquad \theta(t) = \frac{i}{2\pi}\int_{-\infty}^{+\infty} d\omega\, \frac{e^{-i\omega t}}{\omega + i\epsilon} . \tag{1}$$

We also mention the identity

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{x}) , \tag{2}$$

where $\nabla^2 = \partial_i\partial_i$ the three-dimensional Laplace operator.

The Noether current corresponding to a symmetry transformation $\delta\phi^a$ of a Lagrangian density $\mathcal{L}(\phi^a, \partial_\mu\phi^a)$ with fields ϕ^a is given by

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\delta\phi^a . \tag{3}$$

The Minkowski metric in four spacetime dimensions is $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, and a representation of the Dirac matrices is

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} , \tag{4}$$

with $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\vec{\sigma})$. They satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

1. Dirac fields (2 points)

The Dirac Lagrangian is given by

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \ . \tag{5}$$

i) Explain all the symbols and write down the Dirac equation. Show that it implies the Klein-Gordon equation.

ii) Determine the free-particle plane wave solutions of the Dirac equation in the rest frame (in four spacetime dimensions).

2. Yukawa potential (3 points)

Consider the Lagrangian density for a massive vector field in four spacetime dimensions with a current-source J^μ,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu - A_\mu J^\mu \ , \tag{6}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

i) Write down the equations of motion for A_μ. Assume furthermore that the current J^μ is conserved, i.e. $\partial_\mu J^\mu = 0$, and use it to derive from this a constraint on A_μ. Consequently, show that the equation of motion for A^0 is a Klein-Gordon equation with a source J^0.

For J_μ the current of a point charge in the origin with $J^\mu = (e\delta^3(\vec{x}), \vec{J} = 0)$, the Klein-Gordon equation for A_0 is solved by

$$A_0(r) = \frac{e}{4\pi^2 ir}\int_{-\infty}^{+\infty} \frac{k\mathrm{d}k}{k^2 + m^2} e^{ikr} \ , \tag{7}$$

where $r = |\vec{x}| = \sqrt{x^i x^i}$.

ii) Evaluate this integral using contour integration to get an explicit form for $A_0(r)$.

iii) Show that this expression solves the Klein-Gordon equation by explicit differentiation and using (2). Show that as $m \to 0$ you reproduce the Coulomb potential.

3. Non-relativistic field theory (5 points)

Consider the following action of a complex scalar field in three spatial dimensions, based on the Lagrangian

$$L = \int d^3\vec{x} \ \mathcal{L} \ , \qquad \mathcal{L} = i\phi^* \partial_t \phi - \frac{1}{2m}\vec{\nabla}\phi^* \cdot \vec{\nabla}\phi \ . \tag{8}$$

Notice that this Lagrangian contains only a single time derivative ∂_t instead of two in a relativistic theory.

i) Write down the equation of motion for ϕ and ϕ^*, the canonical momenta, and the classical Hamiltonian.

ii) Determine the conserved current associated with the symmetry $\phi \to e^{i\alpha}\phi$. Show that the corresponding charge is given by

$$Q = \int d^3\vec{x}\ \phi^* \phi\ , \tag{9}$$

and is indeed conserved, so time independent.

We now quantize the theory in the Schrödinger picture and expand the field operators in a Fourier decomposition as

$$\phi(\vec{x}) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^{3/2}}\, a(\vec{p})\, e^{i\vec{p}\cdot\vec{x}}\ , \qquad \phi^\dagger(\vec{x}) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^{3/2}}\, a^\dagger(\vec{p})\, e^{-i\vec{p}\cdot\vec{x}}\ . \tag{10}$$

Notice that the normalizations here are chosen a little different than in the course.

iii) Show that the correct commutation relations are given by ($\hbar = 1$)

$$[a(\vec{p}), a^\dagger(\vec{p}')] = \delta^3(\vec{p} - \vec{p}')\ . \tag{11}$$

Why do we have only a single set of creation and annihilation operators even though ϕ is complex and consists of two real fields? What does it have to do with the field being non-relativistic?

iv) Write down the Hamiltonian in terms of a and $a^\dagger$ and show that the one-particle states have the energy of a non-relativistic particle of mass m. Similarly write the charge Q in terms of creation and annihilation operators and show that $[Q, H] = 0$.

v) Determine the time dependence of the quantized fields in the Heisenberg picture and compute the Feynman propagator for the non-relativistic scalar field, defined by

$$D_F(x - y) \equiv \langle 0|T\phi^\dagger(x)\phi(y)|0\rangle\ , \tag{12}$$

where T stands for time ordening and $x = (x^0, \vec{x})$. Write the result first as a three-dimensional integral over momenta $d^3\vec{p}$ and then as a four-dimensional integral

$$D_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{-p_0 + \frac{\vec{p}^2}{2m} + i\epsilon}\, e^{-p\cdot(x-y)i}\ , \tag{13}$$

with $p = (p_0, \vec{p})$.