EXAMINER: DR. UMUT GÜRSOY
DATE: 08/11/2018
TIME: 17:00 - 20:00

UTRECHT UNIVERSITY
MIDTERM EXAM

# Midterm exam for Quantum Field Theory

- Write your **name and student number on every sheet**.
- There are 3 problems. Write your answers to the individual problems on different sheets.
- Make sure that your **answers are understandable and readable**. In doubt, explain with a short comment what you are doing.

## Problem 1: Short questions [30+7pt]

In this problem we ask you some basic questions concerning the lectures. You should give **short** answers. Do not lose too much time on this problem.

(i) **[2pt]** Write down the massive Klein-Gordon equation.

(ii) **[4pt]** What are the connected components of the Lorentz group and how are they related?

(iii) **[4pt]** State the spin-statistics theorem.

(iv) **[4pt]** What is the physical interpretation of the Feynman, retarded and advanced propagators in terms of time-ordering?

(v) **[4pt]** How are forces accounted for in quantum field theory?

(vi) **[4pt]** Consider a Lorentz invariant classical field theory with a given action $S$. How do you obtain the Hamiltonian density from this action? How does the Hamiltonian density transform under a Lorentz transformation?

(vii) **[4pt]** What is the transformation law of the scalar field $\phi(x)$ under the Lorentz transformation $x^\mu \to \bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu$ where $\Lambda$ is a Lorentz matrix? Denoting the transformed field by $\bar{\phi}(\bar{x})$ what is $\bar{\phi}(x) - \phi(x)$ for infinitesimal $\bar{x} - x$?

(viii) **[4pt]** Write down the total 4-momentum of a free relativistic scalar field in terms of creation and annihilation operators.

**Hint**: you should be able to infer this from the expression for the Hamiltonian.

(ix) **Bonus: [4pt]** State Wick's theorem for the free scalar field.

(x) **Bonus: [3pt]** What is the mass dimension of a scalar field in (2+1) dimensions, described by the action

$$S = -\frac{1}{2}\int \mathrm{d}^3x\, \partial_\mu\phi\partial^\mu\phi\,. \tag{1}$$

## Problem 2: Noether's theorem [35pt]

Consider the theory of two free, complex scalar fields $\phi_i$, $i = 1, 2$ described by the Lagrangian density

$$\mathscr{L} = -\delta^{ij}\partial_\mu\phi_i\partial^\mu{\phi_j}^\dagger - m^2\delta^{ij}\phi_i{\phi_j}^\dagger\,, \tag{2}$$

where we employed the "sum convention", i.e. we sum over repeated indices $i, j$.

(i) **[5pt]** Show that the Lagrangian density (2) is invariant under the transformation

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to \begin{pmatrix}\mathrm{e}^{-i\alpha} & 0\\ 0 & \mathrm{e}^{i\alpha}\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\,, \tag{3}$$

where $\alpha$ is a real parameter. The transformations for ${\phi_i}^\dagger$ follow from (3) by hermitian conjugation.

(ii) **[7.5pt]** Compute the Noether current $j^\mu$ associated to the symmetry transformation (3) using the infinitesimal version of the transformation of (3).

**Hint:** The Noether current is given by

$$j^\mu = \frac{\partial\mathscr{L}}{\partial\partial_\mu\Phi^I}\Delta\Phi^I - J^\mu\,, \tag{4}$$

where the index $I$ labels all fields in the Lagrangian (the hermitian conjugate fields are considered independent fields).

We now quantize the complex fields $\phi_i$ in the standard way, i.e. we expand them as

$$\phi_i(x) = \int \widetilde{\mathrm{d}k}\left[a_i(\boldsymbol{k})\,\mathrm{e}^{ik\cdot x} + {b_i}^\dagger(\boldsymbol{k})\,\mathrm{e}^{-ik\cdot x}\right]\,, \qquad \widetilde{\mathrm{d}k} = \frac{\mathrm{d}^3k}{2\omega(2\pi)^3}\,, \tag{5}$$

where the only non-trivial commutators are given by

$$\begin{aligned}[a_i(\boldsymbol{k}), {a_j}^\dagger(\boldsymbol{k}')] &= 2\omega(2\pi)^3\delta_{ij}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\,,\\ [b_i(\boldsymbol{k}), {b_j}^\dagger(\boldsymbol{k}')] &= 2\omega(2\pi)^3\delta_{ij}\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\,.\end{aligned} \tag{6}$$

(iii) **[15pt]** Using (5) and (6) (which you do not have to show), compute the conserved charge $Q$ associated to the Noether current $j^\mu$ in terms of the creation and annihilation operators ${a_i}^{(\dagger)}$ and ${b_i}^{(\dagger)}$.

(iv) **[7.5pt]** How does the charge operator $Q$ transform under a Lorentz transformation $\Lambda$? Show this by a calculation.

## Problem 3: Path integrals with scalar fields [35pt]

Consider the theory of two real scalar fields $\phi_{1,2}$ in (3+1) dimensions described by the Lagrangian density

$$\mathscr{L} = -\frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m^2\phi_1^2 - \frac{1}{2}m^2\phi_2^2 + g\phi_1\phi_2 + J_1\phi_1 + J_2\phi_2\,, \qquad (7)$$

where we already included the sources $J_{1,2}$ for the scalar fields $\phi_{1,2}$.

(i) **[20pt]** Compute the generating functional

$$Z[J_1, J_2] = \int \mathscr{D}\phi_1\mathscr{D}\phi_2 \, \exp\left(i\int \mathrm{d}^4x\,\mathscr{L}\right) \qquad (8)$$

by **explicity** performing the path integral (8). Normalize the generating functional such that $Z[J_1 = 0, J_2 = 0] = 1$.

(ii) **[10pt]** Compute the two-point functions $\langle\phi_i(x_1)\phi_j(x_2)\rangle = \langle 0|T\phi_i(x_1)\phi_j(x_2)|0\rangle$ for $i, j = 1, 2$. If you were unable to solve problem (i) you will get partial credit by computing the correlation functions for the case $g = 0$ from the corresponding generating functional.

(iii) **[5pt]** Argue whether or not this is an interacting theory.