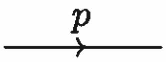
# Final: Quantum Field Theory 2023

**Instructions:**

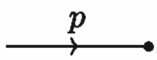
- Write your name and student number on every sheet.
- Make sure your answers are understandable and readable.
- **Please solve each problem on a different sheet.**

**Formularium and conventions:**

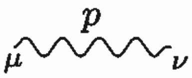
- We use the Minkowski metric in four spacetime dimensions $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$
- Algebra of the Dirac $\gamma-$matrices: $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$
- Hermicity properties of the $\gamma-$matrices: $\gamma^{i\,\dagger} = -\gamma^i, \quad \gamma^{0\,\dagger} = \gamma^0$
- Completness relations for spinors: $\sum_s u^s(p)\bar{u}^s(p) = -\not{p} + m, \quad \sum_s v^s(p)\bar{v}^s(p) = -\not{p} - m$ where $\bar{u} = u^\dagger\gamma^0$ and $\not{p} = \gamma^\mu p_\mu$.
- Feynman rules for QED

Spinors : 

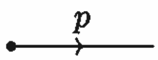
$= \frac{-i(-\not{p}+m)}{p^2+m^2-i\epsilon}$



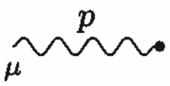
$= \bar{u}^s(p)$ outgoing electron

Photons : 

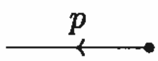
$= \frac{-i\,\eta_{\mu\nu}}{p^2-i\epsilon}$



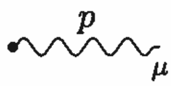
$= u^s(p)$ incoming electron



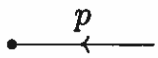
$= \epsilon_\mu(p)$ incoming photon



$= \bar{v}^s(p)$ incoming positron



$= \epsilon^*_\mu(p)$ outgoing photon



$= v^s(p)$ outgoing positron

(Dots denote insertion of the source where the external particles are connected.)

**Exercise 1: Short questions [20+2 points]** In this problem, we ask some short questions to test your general knowledge. You do not need to do any calculation to answer. Do not spend too much time with these questions and concise answers are welcome.

(i) **[5 pts]** Explain what helicity and chirality of a Dirac spinor are. Are they conserved under the time evolution?

(ii) **[5 pts]** How many linearly independent polarization vectors do on-shell massive and massless vector fields have?

(iii) **[5 pts]** What is the interaction term that couples fermions to the gauge field in the QED Lagrangian? Why does not this term violate gauge invariance in the action?

(iv) **[5 pts]** Consider the $(1/2, 1/2)$ representation of Lorentz group, what do these numbers correspond to? How many degrees of freedom does this representation have?

**[Bonus: 2 pts]** What are the spin eigenvalues in this representation?

## Exercise 2: Feynman diagrams of $\phi^3$-Yukawa theory [30+6 points]

Consider the theory of a scalar $\phi$ and spinor $\psi$ interacting through the following cubic interactions:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_\phi^2\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m_\psi)\psi - \frac{\lambda}{3!}\phi^3 - g\bar{\psi}\phi\psi\,. \tag{1}$$

1. **[10 pts]** What is the full equation of motion of scalar $\phi$ including contributions from the interaction terms?

2. **[10 pts]** Write down all Feynman rules of this theory in momentum space. Use dashed lines for scalars and solid lines for spinors.

3. **[10 pts]** Draw all inequivalent, connected, tree-level Feynman diagrams of process $\phi+\psi \to \phi+\psi$. (Hint: There are 3 diagrams. If you draw more, you will lose points.)

4. **[Bonus: 6 pts]** Draw 4 inequivalent, connected, 1-loop level Feynman diagrams for the process $\phi+\psi \to \phi+\psi$. At least one of them should be of $\mathcal{O}(\lambda^2 g^2)$.

## Exercise 3: Bhabha scattering amplitude [35+5 points]

The Bhabha scattering is an elementary process in QED that involves an electron and a positron in the following way:

$$e^+ + e^- \to e^+ + e^-$$

(i) **[10 pts]** Draw all the topologically distinct tree-level diagrams.

(ii) **[10 pts]** Using the Feynman rules for QED, write the scattering amplitude $T$ for each of the graphs.

(iii) **[5 pts]** From now on consider only the scattering amplitude of the s-channel diagram $T_{s-ch}$. [Hint: s-channel is the one where the internal propagator carries the momentum equal to sum of the two incoming particles.] Compute the conjugate amplitude $T^\dagger_{s-ch}$ and the amplitude square $|T_{s-ch}|^2$.

In the scattering amplitude you just computed, the polarization states of both initial and final fermions are specified. However most of the time we are interested in an unpolarized scattering, which is obtained by averaging over all initial polarizations states and summing over the final polarization states. In our case this translates into computing:

$$\frac{1}{4}\sum_{spins} |T|^2 \tag{2}$$

where we are summing over the spins of the initial and final states (4 in total). This computation can be done in two steps:

(iv) **[10 pts]** Show that the following relation holds:

$$\sum_{s}\sum_{s'}[\bar{u}^{s'}_a(p)\gamma^\mu_{ab}v^s_b(k)][\bar{v}^s_c(k)\gamma^\nu_{cd}u^{s'}_d(p)] = \text{Tr}[(-\not{p}+m)\gamma^\mu(-\not{k}-m)\gamma^\nu]$$

where Latin and Greek letters denote spinorial and Lorentz indices, respectively. Remember that $\text{Tr}[ABC] = A_{ab}B_{bc}C_{ca}$, that the trace is invariant under cyclic permutations ($\text{Tr}[ABC] = \text{Tr}[BCA] = \text{Tr}[CAB]$) and it is linear ($\text{Tr}[A+B] = \text{Tr}[A] + \text{Tr}[B]$).

(v) **[Bonus: 5 pts]** Exploiting now the relation derived in point (iv), evaluate the unpolarized scattering amplitude (2) for $T_{s-ch}$, i.e. write the final result for the unpolarized scattering amplitude as a function of the initial and final momenta and the mass of the particles only.
You will need the following identities: $\text{Tr}[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu] = 4(\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\mu\nu})$, $\text{Tr}[\gamma^\alpha\gamma^\mu\gamma^\beta] = 0$ and $\text{Tr}[\gamma^\mu\gamma^\nu] = 4\eta^{\mu\nu}$.

## Exercise 4: Chern-Simons Theory Coupled to Matter [15 + 5 points]

Consider the 2+1 dimensional theory of Dirac spinor, $\psi$, and vector field, $A^\mu$, described by Lagrangian density

$$\mathcal{L} = i\overline{\psi}\gamma^\mu D_\mu \psi - m\overline{\psi}\psi - \frac{k}{4\pi}\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \tag{3}$$

where $D_\mu = \partial_\mu - ieA_\mu$ and $\epsilon^{\mu\nu\rho}$ is totally antisymmetric Levi-Civita symbol, and $m$, $k$ and $e$ are real numbers.

(i) **[5 pts]** Derive the equations of motion for the fields $\psi$ and $A^\mu$ (including the interaction terms).

(ii) **[10 pts]** Under local $U(1)$ action parameterized by $\alpha(x)$, the spinor transforms as $\psi \to e^{i\alpha(x)}\psi$. Show that the corresponding action is invariant under this local U(1) transformation provided that the vector field $A^\mu$ transforms as $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha$.

From now on, treat $A^\mu$ as a gauge field that transforms as you have derived above.

(iii) **[Bonus: 5 pts]** Find the Noether current ($J^\mu$) and charge (Q) associated with the $U(1)$ local symmetry and give the physical interpretation of the conserved charge.