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In elektronische vorm beschikbaar gemaakt door de $\mathcal{TBC}$ van A–Eskwadraat.
Het college NS-TP401 werd in 2005/2006 gegeven door B. de Wit.

Quantum Field Theory (NS-TP401)
02 februari 2006

Opgave 1. Maxwell theory in $1+1$ space-time dimensions

Consider the action of Maxwell theory in two spacetime dimensions,

$$S[A] = \int dxdt(\tfrac{1}{2}F_{tx}^2 + \theta F_{tx} + J^\mu A_\mu) \tag{1}$$

Here F_{tx} is the field strenght defined by $F_{tx} = \partial_t A_x - \partial_x A_\mu$, where the two-vector $A_\mu = (A_t, A_x)$ comprised the potentials, and $J^\mu = (J^t, J^x)$ is some external current. Note that, in two space-time dimensions, there exists no magnetic field whereas the electric field equals $E = -F_{tx}$. The term proportional to the constant θ has no analogue in higher dimensions (at least, not in a Lorentz invariant setting).

a) Prove that the electric field is invariant under the gauge transformations of the form $\delta A_\mu = \partial_\mu \Lambda(t,x)$. Derive the field equations for A_μ and write them as first-order equations for E. Show that current J^μ must be conserved. In the absence of the current, give the solutions of E.

b) Use the gauge transformations to set $A_t(t,x) = 0$. Give the gauge parameter $\Lambda(t,x)$ that is required for this, expressed as an integral over A_t. Write down the resulting Lagrangian in the $A_t = 0$ gauge, which depends only on A_x. Observe that we still have a residual invariance under gauge tranformations with functions $\Lambda(x)$ that depend only on x and no longer t.

c) Write down the field equations in this gauge and note there is one field equation less in this case. For the moment ignore this equation which will have to be imposed eventually as the so-called Gauss constraint. Write down the canonical momentum $\pi(t,x)$ associated to $A_x(t,x)$ and show how it is related to $E(t,x)$.

d) Write down the canonical commutation relations for $A_x(x)$ and $\pi(y)$ (in the Schrödinger picture, so that we suppress the time dependence).

e) Write down the Hamiltonian and define the wave function in the 'coordinate' representation. Here and henceforth suppress the external current J^μ. Give the form of the momentum in this representation. What is the lowest-energy state?

f) Let us now return to the Gauss constraint. Show that, for arbitrary functions $\Lambda(x)$, $Q[\Lambda(x)] = \int \mathrm{d}x \Lambda(x) \partial_x \pi(x)$ vanishes classically but not as an operator.

g) **Bonus question**: Consider $Q[\Lambda]$ as an operator and calculate the commutator $[Q, A_x(y)]$. Interpret the result. Argue now that physical wave functions should be annihilated by the operator Q. What are the physically relevant wavefunctions?

Opgave 2. Feynman diagrams for fermions

Consider a field theory for a massive complex fermion field $\psi(t,\vec{x})$ in three space-time dimensions (with c, the velocity of light, equal to $c=1$), described by the Lagrangian,

$$\mathcal{L} = i\psi^\dagger \partial_t \psi + \psi^\dagger \sigma_3 \vec{\sigma} \cdot \vec{\nabla} \psi - m\psi^\dagger \sigma_3 \psi. \tag{2}$$

Here σ_i are the Pauli matrices satisfying $\sigma_i \sigma_j = \delta_{ij}\mathbf{1} + i\varepsilon_{ijk}\sigma_k$, and conventionally defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that ψ is thus a spinor with two components. The arrow denotes vectors in the two spatial dimensions, *i.e.*,

$$\vec{x} = (x_1, x_2), \vec{p} = (p_1, p_2), \vec{\nabla} = (\partial_1, \partial_2), \vec{\sigma} = (\sigma_1, \sigma_2). \tag{3}$$

a) Derive the field equations for $\psi(t, \vec{x})$ and $\psi^\dagger(t, \vec{x})$.

b) We now introduce a scalar commuting field ϕ which interacts with the fermions. The additional Lagrangian equals

$$\mathcal{L}_{\text{add}} = -\tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2}\mu^2\phi^2 + g\phi\psi^\dagger\sigma_3\psi. \tag{4}$$

The presence of the matrix σ_3 is required by three-dimensional Lorentx invariance. Write the quadratic part of the fermionic and bosonic actions as,

$$\frac{i}{\hbar}S_0[\phi, \psi, \psi^\dagger] = \int \mathrm{d}^3x\, \mathrm{d}^3y \left\{ -\tfrac{1}{2}\phi(x)\Delta_\phi^{-1}(x,y)\phi(y) - \psi^\dagger(x)\sigma_3\Delta_\psi^{-1}(x,y)\psi(y) \right\}. \tag{5}$$

We know that $\Delta_\phi^{-1}(x,y) = \Delta_\phi^{-1}(x-y) = \frac{i}{\hbar}[-\partial_\mu^2 + \mu^2]\delta^3(x-y)$. Determine now also $\Delta_\psi^{-1}(x,y)$, which takes the form of a 2×2 matrix, in a similar form. Also write down the differential equations satisfied by $\Delta_\phi(x)$ and $\Delta_\psi(x)$.

c) Write down the path integral for both the free bosonic and free fermionic action, with external source terms given by,

$$\int d^3x \{J_\phi(x)\phi(x) + \psi^\dagger(x)\sigma_3 J_\psi(x) + J_\psi^\dagger(x)\sigma_3\psi(x)\}, \tag{6}$$

which should be added to (5). Note that the source J_ψ is a two-dimensional spinor.

d) Calculate the path integral based on the free action with source terms, ignoring the determinants generated by the (Gaussian) integration over the various fields. This result can be used to calculate Feynman diagrams in the way that was explained in class.

e) Using the previous result derive the two-point functions in tree approximation,

$$\begin{aligned} \langle\phi(x)\phi(y)\rangle &= \frac{\int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\psi^\dagger\,\phi(x)\phi(y)e^{\frac{i}{\hbar}S[\phi,\psi,\psi^\dagger]}}{\int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\psi^\dagger e^{\frac{i}{\hbar}S[\phi,\psi,\psi^\dagger]}}, \\ \langle\psi_\alpha(x)\psi_\beta^\dagger(y)\rangle &= \frac{\int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\psi^\dagger\,\phi_\alpha(x)\psi_\beta^\dagger(y)e^{\frac{i}{\hbar}S[\phi,\psi,\psi^\dagger]}}{\int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\psi^\dagger e^{\frac{i}{\hbar}S[\phi,\psi,\psi^\dagger]}}, \end{aligned} \tag{7}$$

where α and β denote the spinor components.

f) Subsequently, derive the expression for the one-point function,

$$\langle\phi(x)\rangle = \frac{\int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\psi^\dagger\,\phi(x)e^{\frac{i}{\hbar}S[\phi,\psi,\psi^\dagger]}}{\int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\psi^\dagger e^{\frac{i}{\hbar}S[\phi,\psi,\psi^\dagger]}}, \tag{8}$$

in the one-loop approximation (which is linear in the coupling constant g).

Opgave 3. The large-N limit

Consider the action for N real scalar fiels ϕ_i $(i = 1 \ldots N)$ and one real scalar field σ, given by

$$S[\phi_i, \sigma] = \int \mathrm{d}^4x \left\{ -\tfrac{1}{2}\sum_i(\partial_\mu\phi_i)^2 - \tfrac{1}{2}m^2\sum_i \phi_i^2 + \sigma\sum_i \phi_i^2 + \tfrac{1}{2}c\sigma^2 \right\}. \tag{9}$$

a) Give the expressions for the propagators and the veritces of the action.

b) Write down, in the one-loop approximation, the expressions for the two selfenergy diagrams for the fields ϕ_i and the one selfenergy diagram for the field σ (do not evaluate the corresponding momentum integrals).

c) Calculate the one-loop diagram with a single external σ-line. Express the result into the (divergent) momentum integral

$$T(m^2) = \int \frac{\mathrm{d}^4 p}{i(2\pi)^4} \frac{1}{p^2 + m^2}. \tag{10}$$

How does the result depend on N?

d) We are interested in the (connected) correlation functions of the fields ϕ_i. To that order introduce a source term J_i for every field ϕ_i (but no source for the field σ). The relevant connected correclation functions are then given by

$$\langle \phi_{i_1}(x_1) \ldots \phi_{i_n}(x_n) \rangle = \frac{\delta}{\delta J_{i_1}(x_1)} \cdots \frac{\delta}{\delta J_{i_n}(x_n)} \log W[J_i] \bigg|_{J_i=0}, \tag{11}$$

where W denotes the full path integral in the presence of the external sources. Determine the value of c such that the theory is equivalent to the one given by an action without the field σ, but with a four-point coupling $\left(\sum_i \phi_i^2\right)^2$ with coupling constant $-g/N$.

Subsequently we assume that c is equal to this special value that you found in d). In the limit of large N with g constant the ϕ^4 coupling vanishes, so that theory is free. We analyze this below to all orders in pertubation theory for all connected correlation functions (11).

e) First consider all diagrams with only external ϕ-lines (coupled via internal σ-lines) but *without* loops formed *exclusively* by ϕ-propagators. In that case, show that only free ϕ-propagators survive in the limit $N \to \infty$.

f) Add a loop consisting exclusively of internal ϕ-propagators. Hence only its external lines are associated with σ, so that this loop must couple through σ-lines to the rest of the diagram. Analyze which diagrams will contribute in the limit $N \to \infty$. Generalize this argument to several ϕ-loops and prove that, in the limit $N \to \infty$, only the two-point correlation functions (11) are nonzero. The theory therefore behaves as a free field theory in this limit. Nevertheless, the two-point function does receive finite contributions in the large-N limit.

g) **Bonus question**: Prove, in the $N \to \infty$ limit, that the quantum corrections only give rise to changes in the ϕ-mass. Denote this modified mass by M. Show that M satisfies the following equation,

$$M^2 = m^2 + 4gT(M^2). \tag{12}$$

Do this by first expanding the right-hand side to g with the aid of the one-loop result. Subsequently, that that both sides of the equation describe the same diagrams.