Instituut voor Theoretische Fysica
Universiteit Utrecht

RETAKE EXAM Quantum Field Theory

Thursday, March 20, 2008, 14.00 - 17.00, BBL 107A

1) Start every exercise on a separate sheet.
2) Write on each sheet: your name and initials. In addition, write on the *first* sheet: your address, postal code and your field of study. Indicate whether you follow the master's programme in theoretical physics.
3) Write legibly and clear! If not, your work will not be graded.
4) The exam consists of **three** exercises.

1. Feynman diagrams and auxiliary fields

Consider a field theory in four space-time dimensions with two real fields, ϕ en A, described by the Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\,\phi^2 - \lambda\,\phi^4 - \tfrac{1}{2}A^2 + A(\mu^2 + g\,\phi^2)\,.$$

i) Give the propagators and vertices. Determine the dimension of the fields and of the coupling-constant and mass parameters λ, g, μ^2 and m^2.

ii) Calculate the self-energy diagrams in the tree approximation and give the masses for the *physical* particles described in this approximation. Subsequently give the full propagators in the tree approximation and use these in the next three questions.

iii) Calculate the (three) self-energy diagrams for the field ϕ in the one-loop approximation. Give the mass-shift of ϕ in that approximation. (Note: do *not* try to evaluate the integrals.) For which value of g does the mass shift vanish?

iv) Calculate the mass shift for the field A in the one-loop approximation. What do you conclude?

v) Solve the equations of motion for A en substitute the result into the Lagrangian, which will then depend only on ϕ. Show that this corresponds to integrating out the field A in the path integral.

vi) Evaluate now again the mass of the field ϕ in tree approximation and compare the result with that of question ii) above.

vii) **Bonus question**: Calculate again the self-energy diagrams in the one-loop approximation? Compare the result with that obtained in question iii).

2. The Witten index

Consider a quantum-mechanical model with a Hamiltonian that depends on a coordinate operator q and a momentum operator p, satisfying the standard commutation relation $[q,p] = \mathrm{i}\hbar$. The Hamiltonian takes the form of a two-by-two diagonal matrix, whose matrix elements are denoted by $H^{\pm}(p,q)$.

i) The class of Hamiltonians that we consider is encoded in functions $W(q)$, and we have

$$H^{\pm}(p,q) = \frac{p^2}{2m} + \tfrac{1}{2} m W^2(q) \pm \tfrac{1}{2}\hbar W'(q)\,, \qquad W' = \frac{\partial W}{\partial q}\,. \tag{1}$$

Write down a function $W(q)$ such that you obtain the Hamiltonian for the supersymmetric harmonic oscillator that was treated in class. Can you give the eigenvalues of this Hamiltonian?

To appreciate why this class of Hamiltonians is special, consider the operators,

$$Q = \frac{1}{\sqrt{2\,m}} \begin{pmatrix} 0 & mW(q) + \mathrm{i}p \\ 0 & 0 \end{pmatrix}, \qquad Q^{\dagger} = \frac{1}{\sqrt{2\,m}} \begin{pmatrix} 0 & 0 \\ mW(q) - \mathrm{i}p & 0 \end{pmatrix}, \tag{2}$$

Furthermore there is an operator $\mathbf{F} = \mathrm{diag}(1,-1)$, which obviously commutes with the Hamiltonian. This operator will be called *fermion* operator: states with eigenvalue $+1$ will be called bosonic and are described by H^{+}; states with eigenvalue -1 will be called fermionic and are described by H^{-}. Of course, these names are a matter of convention and can be interchanged.

ii) Show that the following operator relations hold,

$$\{Q, Q^{\dagger}\} = Q\,Q^{\dagger} + Q^{\dagger} Q = H\,, \qquad Q^2 = (Q^{\dagger})^2 = 0\,. \tag{3}$$

Prove from these relations that Q and $Q^{\dagger}$ commute with the Hamiltonian. Hence they define conserved charges. Furthermore, prove that $\mathbf{F}$ anticommutes with the operators Q and $Q^{\dagger}$, so that these operators may be regarded as fermionic.

iii) Prove from the first (anticomutation) relation (3), that the Hamiltonian must have non-negative expectation values for any state $|\psi\rangle$. Therefore we expect an infinite number of postive-energy states and a finite number of zero-energy states. Show that the latter must satisfy $Q|\psi\rangle = Q^{\dagger}|\psi\rangle = 0$.

In this problem we are interested in counting the number of zero-energy states. One can show, using the above properties, that the positve-energy eigenstates appear always in pairs consisting of a bosonic and a fermionic state. Therefore it is interesting to consider the following traces over the full Hilbert space, $\mathrm{Tr}[\mathrm{e}^{-\beta H}]$ and $\mathrm{Tr}[\mathbf{F}\mathrm{e}^{-\beta H}]$. The first expression is the partition function, which is a very complicated function of β, depending on the function $W(q)$. The second expression equals the difference of the partition function for the bosonic states minus the partition function for the fermionic states.

iv) Argue that $\mathrm{Tr}[\mathbf{F}\mathrm{e}^{-\beta H}]$ receives contributions from only zero-energy states. Therefore it must be independent of β and furthermore it must be equal to an integer. What is the meaning of this integer? Expressions that always take integer values are often called 'index' in the mathematical literature. The expression at hand is known as the Witten index.

We now consider the field-theoretic representation of the above. The Hamiltonian (1) corresponds to a Lagrangian with one bosonic and one fermionic coordinate. The bosonic (Euclidean) action takes the form

$$S_{\text{bosonic}} = \int_0^{\hbar\beta} \mathrm{d}\tau \, \left[\tfrac{1}{2} m^2 \dot{q}^2 + \tfrac{1}{2} m \, W^2(q)\right] , \tag{4}$$

where we suppressed a term of order $\hbar$. The fermionic action reads,

$$S_{\text{fermionic}} = \hbar \int_0^{\hbar\beta} \mathrm{d}\tau \, [\bar{\alpha}(\tau)\dot{\alpha}(\tau) - \bar{\alpha}(\tau) \, W'(q(\tau)) \, \alpha(\tau)] , \tag{5}$$

where we note the proportionality factor $\hbar$. To evaluate the partition function one evaluates the path-integral over periodic bosonic and anti-periodic fermionic paths. (The standard fermionic boundary term in the fermionic action is required for deriving this result, so that it was suppressed in (5).) To obtain the Witten index, one evaluates the path integral but now with periodic boundary conditions for *both* bosonic and fermionic fields.

Since we have established that the expression does not depend on β, we will evaluate the path integral in the limit $\hbar\beta \to 0$. The paths can then be taken constant over the $(0, \hbar\beta)$ interval. In class we have followed this approach for an particle moving in some potential $V(q)$, and we derived the following (bosonic) result,

$$Z_\beta \approx \sqrt{\frac{m}{2\pi\hbar^2\beta}} \int_{-\infty}^{\infty} \mathrm{d}q \, \mathrm{e}^{-\beta V(q)} . \tag{6}$$

To include the fermions, we should simply include their contribution into the potential $V(q)$ and subsequently integrate over $\bar{\alpha}$ and α (because the τ-dependence disappears in the $\hbar\beta \to 0$ limit, there are only two fermionic variables left).

v) Show that the expression for the Witten index in the limit described above, amounts to substituting $V(q) = \frac{1}{2} m W^2(q) - \hbar\bar{\alpha} \, \alpha \, W'(q)]$ in (6), subject to an overall integral $\int \mathrm{d}\bar{\alpha} \, \mathrm{d}\alpha$. Perform the latter fermionic integrals and show that the answer becomes independent of $\hbar$, but not (yet) of β.

vi) By means of a variable change, prove that the resulting bosonic integral equals -1. **Bonus question**: Scrutinize the variable substitution to argue that other values remain possible.

3. Kaluza-Klein compactification of Maxwell theory

Consider the action of Maxwell theory in five space-time dimensions,

$$S[A_M] = -\frac{1}{4\,g^2}\int \mathrm{d}^5x\; F_{MN}\,F^{MN}\,, \tag{7}$$

where $M, N = 0, 1, 2, 3, 4$ and $F_{MN} = \partial_M A_N - \partial_N A_M$. The action is invariant under gauge transformations,

$$A_M \to A_M + \partial_M \Lambda(x)\,, \tag{8}$$

where Λ is an arbitrary function of the coordinates x^M.

We will be considering the situation where one of the coordinates, $x^4 \equiv y$, is compactified on a circle with radius R, so that $x^M = (x^\mu, y)$ with $\mu = 0, 1, 2, 3$ and $0 \leq y < 2\pi R$. The (real) gauge field A_M is decomposed accordingly,

$$A_M = (A_\mu, \phi)\,, \quad \text{where} \quad \phi \equiv A_4\,. \tag{9}$$

Both fields, A_μ and ϕ, depend on both x^μ and y. In view of the fact that y parametrizes a circle, we assume the fields to be periodic in y. Therefore they can be written in terms of a Fourier series, leading to an infinite number of four-dimensional fields, $A_\mu^{(n)}(x)$ and $\phi^{(n)}(x)$, labeled by integer numbers n. In the following, x will always refer to the four-dimensional coordinates.

i) Write down this Fourier series. Which of these four-dimensional fields are real and which ones are complex. Assuming that there are complex fields, what is the consequence of the reality condition on the original field A_M?

ii) For the moment, assume that $\Lambda(x, y)$ is also periodic in y, so that it can be written as a similar Fourier sum, leading to infinitely many parameters $\Lambda^{(n)}(x)$, characterized again by integers n. Write down the gauge transformations for A_μ and ϕ. In the four-dimensional context we are thus confronted with an infinite number of fields and gauge transformations. Prove that we can use the gauge transformations associated with $\Lambda^{(n)}(x)$ to put the fields $\phi^{(n)}(x)$ to zero, provided that $n \neq 0$.

Henceforth we will thus suppress the fields $\phi^{(n)}(x)$ with $n \neq 0$. Only the gauge transformations associated with the function $\Lambda^{(0)}(x)$ remain.

iii) Substitute the Fourier series for A_μ and ϕ into the Lagrangian (7), which is then expressed in terms of the fields $A_\mu^{(n)}(x)$ and $\phi^{(0)}(x)$. First use that,

$$F_{\mu 4} = -F_{4\mu} = \partial_\mu \phi - \partial_y A_\mu\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{10}$$

which are to be decomposed in Fourier modes, subject to the gauge condition discussed in ii). The result is an infinite sum of free-field Lagrangians in four space-time dimensions. Consider the spectrum. Which four-dimensional fields are massless and what is their spin? Which fields are massive, what is their spin and what is their mass. Do all fields have different masses, or are there certain mass degeneracies?

iv) Let us now reconsider the gauge transformation parameter $\Lambda(x, y)$ and try to understand whether we can relax the periodicity condition. Clearly, the fields have to remain periodic on the circle, and therefore the derivatives $\partial_\mu \Lambda$ and $\partial_y \Lambda$ must be taken periodic in y. Consider a Fourier sum for these derivatives and show that there is one particular set of functions for which Λ is not periodic, although its derivatives are periodic. (Use that a function that is not periodic over a certain interval, can always be written as a linear function plus a periodic function).

v) Based on the previous question can you now indicate which symmetries are implied by the higher-dimensional gauge transformations for the Lagrangian derived in question iii)?

vi) Demonstrate that the Lagrangian obtained in iii) does indeed exhibit all these symmetries.

vii) **Bonus question**: Assume that in five space-time dimensions there is also a complex scalar field Φ with 'charge' q. Under the original gauge transformations, this field thus transforms as

$$\Phi(x, y) \to \mathrm{e}^{\mathrm{i}q\Lambda(x,y)}\, \Phi(x, y)\,. \tag{11}$$

Assume that Φ is periodic over the circle parametrized by y and derive the consequences for $\Lambda(x, y + 2\pi R) - \Lambda(x, y)$. Consider again a Fourier decomposition for $\Lambda(x, y)$ and compare the result with the results obtained in v). Show that the results agree in the limit where R is shrunk to zero.