Instituut voor Theoretische Fysica
Universiteit Utrecht

FINAL EXAM Quantum Field Theory

Thursday, January 31, 2008, 14.00 - 17.00, BBL 105B

1) Start every exercise on a separate sheet.
2) Write on each sheet: your name and initials. In addition, write on the *first* sheet: your address, postal code and your field of study. Indicate whether you follow the master's programme in theoretical physics.
3) Please write legibly and clear.
4) The exam consists of **three** exercises.

1. Fermionic and bosonic harmonic oscillators

In the lectures we found the following discretized version of the fermionic path integral, which we present here in the Euclidian setting,

$$W(\tau_N,\bar\alpha_N;\tau_0,\alpha_0) = \prod_{i=1}^{N-1}\int \mathrm{d}\bar\alpha_i\,\mathrm{d}\alpha_i \\ \times\exp\left(\bar\alpha_N\,\alpha_N - \sum_{i=0}^{N-1}\Big[\bar\alpha_{i+1}(\alpha_{i+1}-\alpha_i)+\frac{1}{\hbar}\Delta H(\bar\alpha_{i+1},\alpha_i)\Big]\right) \tag{1}$$

For the fermionic harmonic oscillator we choose $H(\bar\alpha_{i+1},\alpha_i)=\hbar[\omega\,\bar\alpha_{i+1}\alpha_i-\omega_0]$, where ω_0 represents a zero-point energy, to be discussed later. The continuum version of the integral (1) reads (we take the limit $\Delta\to 0$, keeping $\tau_N-\tau_0=N\Delta$ finite in the usual fashion),

$$W=\int\mathcal{D}\bar\alpha(\tau)\mathcal{D}\alpha(\tau)\;\exp\left(\bar\alpha(\tau_2)\alpha(\tau_2)-\int_{\tau_1}^{\tau_2}\mathrm{d}\tau\,\Big[\bar\alpha\dot\alpha+\hbar^{-1}H(\bar\alpha,\alpha)\Big]\right)\,, \tag{2}$$

where $H(\bar\alpha,\alpha)=\hbar[\omega\,\bar\alpha(\tau)\alpha(\tau)-\omega_0]$ and τ_1 and τ_2 correspond to τ_0 and τ_N, respectively, in the discretized expression (1). We are interested in evaluating the path integral by using the semiclassical approximation, which is exact in this case.

i) Write down the field equations for $\alpha(\tau)$ and $\bar\alpha(\tau)$ and evaluate their classical solutions expressed in the proper boundary values at τ_1 and τ_2.
ii) Write down the integrand that appears in the path integral (2) for these classical solutions.

iii) As always the result of the semiclassical approximation is expressed in terms of the integrand taken at the classical solution times a gaussian integral involving the fluctuations around this solution. To determine the gaussian integral, return to the discretized version of the path integral (1) and substitute the required boundary values for the fields. Prove that this path integral is equal to 1.

iv) Combine the above results to write down the resulting expression for the path integral W and show that it satisfies the product rule, thus confirming the normalization derived in iii). Furthermore evaluate the trace and the supertrace of W from the corresponding integrals over the fermionic variables, using $\beta\hbar = \tau_2 - \tau_1$. These will yield the expression for the partition function $\mathrm{Tr}[\mathrm{e}^{-\beta H}]$ and for $\mathrm{Tr}[(-)^F \mathrm{e}^{-\beta H}]$, respectively. Here F is the fermion number operator, which equals 0 for a bosonic and 1 for a fermionic state. Explain the result for the partition function in terms of the ground state and the excited states of a fermionic harmonic oscillator.

v) Let us now also introduce a bosonic harmonic oscillator into the path integral. We know that the corresponding partition sum takes the form,

$$Z^{\mathrm{boson}}(\beta) = \frac{\mathrm{e}^{-\beta\hbar\omega/2}}{1 - \mathrm{e}^{-\beta\hbar\omega}} \,. \tag{3}$$

Combine this result with the expression for the supertrace of the fermionic harmonic oscillator and fix the value for the zero-point energy ω_0 such that the supertrace becomes independent of β.

vi) Can you explain the β-independence of the supertrace? Assume that, for a similar theory with a more general potential, the result for the supertrace would also be β-independent. Can you deduce in that case why the supertrace should take an integer value?

2. Renormalization of Yukawa couplings

Consider a field theory for a massive complex fermion field $\psi(x)$ and a scalar field $\phi(x)$ in $d = 4$ space-time dimensions. The free Lagrangian equals,

$$\mathcal{L}_0 = -\bar\psi(\partial\!\!\!/ + M)\psi - \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 \,. \tag{4}$$

We consider two types of interactions between these fields,

$$\mathcal{L}_1 = g_1\, \bar\psi\psi\, \phi \,, \qquad \mathcal{L}_2 = g_2\, \bar\psi\gamma^\mu\psi\, \partial_\mu\phi \,. \tag{5}$$

i) Write down the Feynman rules for the two theories given by $\mathcal{L}_0 + \mathcal{L}_1$ and $\mathcal{L}_0 + \mathcal{L}_2$. Explain what happens with the two Lagrangians when shifting the field ϕ with a constant.

ii) Consider the one-loop diagrams contributing to the fermion self-energy and give the explicit expressions for the two theories. Are the diagrams divergent and, if so, indicate the counterterms that one needs to absorb all the divergencies.

iii) What are the mass dimensions of the fields and the coupling constants in the two theories? Are these theories renormalizable by power counting and why (not)?

iv) Which of these two theories are both renormalizable by power counting and strictly renormalizable? Explain your answer.

v) Answer these last questions iii) and iv) again, but now for $d = 2$. Can you argue that the theory based on $\mathcal{L}_1$ is in fact super-renormalizable? **Hint**: consider carefully how possible counterterms can be absorbed into the parameters *and* into the fields of the initial Lagrangian.

3. Unstable particle

Consider the action for two scalar fields, ϕ and σ, described by the Lagrangian,

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{2}(\partial_\mu\sigma)^2 - \tfrac{1}{2}M^2\sigma^2 + g\,\sigma\,\phi^2\,. \tag{6}$$

i) Give the expressions for the propagators and vertices of this Lagrangian. Include the iϵ-terms in the propagators.

ii) Compute the amplitude for two incoming and two outgoing ϕ-particles in tree approximation, restricting the values of the momenta assigned to the external lines by $p^2 = -m^2$. Assuming that the internal σ lines carry momenta q such that $q^2 + M^2$ is large, one can safely drop the iϵ-terms in the propagators. Note, however, that this will therefore depend on the values taken by the momenta of the external lines.

iii) Assuming that $M > 2m$, the particle associated with σ is not stable and can decay into two ϕ-particles. This should imply that its propagator $\Delta_\sigma(q^2)$ will no longer exhibit a pole at some real value of q^2. In lowest order this is not the case, as there is a pole at $q^2 = -M^2$. Therefore it is relevant to see what happens when including a one-loop self-energy diagram with two external σ-lines and two internal ϕ-lines. The shift in the inverse propagator, according to Dyson's equation, takes the form $q^2 + M^2 - [\mathrm{i}(2\pi)^4]^{-1}\Sigma_\sigma(q^2)$, where $\Sigma_\sigma(q^2)$ represents the contribution of the σ-self-energy diagram. Prove that this diagram can be written as,

$$\frac{1}{\mathrm{i}(2\pi)^4}\Sigma_\sigma(q^2) = 2\mathrm{i}g^2\int \mathrm{d}^4x\;\Delta_\phi(x)\,\Delta_\phi(x)\,\mathrm{e}^{-\mathrm{i}q\cdot x}\,. \tag{7}$$

iv) To handle the iϵ-terms in $\Delta_\phi(x)$ we will not convert the expression (7) directly into the momentum representation. Instead we first use an identity for the propagator Δ_ϕ, which is a generalization of an identity that we have discussed in the lectures for the harmonic oscillator,

$$\begin{aligned}\Delta_\phi(x) &= \frac{1}{\mathrm{i}(2\pi)^4}\int \mathrm{d}^4p\;\frac{\mathrm{e}^{\mathrm{i}p\cdot x}}{p^2+m^2-\mathrm{i}\epsilon} \\ &= \theta(x^0)\,\Delta^+(x) + \theta(-x^0)\,\Delta^-(x)\,. \end{aligned}\tag{8}$$

Here $\Delta^{\pm}(x)$ is defined by

$$\Delta^+(x) = [\Delta^-(x)]^* = \Delta^-(-x) = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3p}{2\,\omega(\mathbf{p})}\;\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}-\mathrm{i}\omega(\mathbf{p})\,x^0}\,, \tag{9}$$

where $\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2+m^2}$.
Using the relations between Δ^+ and Δ^- indicated in equation (9), prove that the imaginary part of (7) can be written as

$$\mathrm{Im}\left[\frac{1}{\mathrm{i}(2\pi)^4}\Sigma_\sigma(q^2)\right] = g^2\int \mathrm{d}^4x\,\left[\Delta^+(x)\,\Delta^+(x)+\Delta^-(x)\,\Delta^-(x)\right]\mathrm{e}^{-\mathrm{i}q\cdot x}\,. \tag{10}$$

v) Write (10) in terms of momenta by substituting (9). Show that the result is positive and will only differ from zero whenever $q^2 < -4\,m^2$. **Hint**: the latter is conveniently verified in the Lorentz frame where $\mathbf{q} = \mathbf{0}$.