GR re-take exam March the 19th, 2010, 9-12 am

The use of auxilary material such as notes, books, calculators, laptops, etc. is not allowed. In the problems below units are used such that the constant c in Einsteins special relativity is equal 1 except in problem 3. Each of the 10 (sub)-questions carry the same weight.

(1) Let $g_{\mu\nu}$ denote the metric tensor. How does it transform under a general coordinate transformation $x^\mu \rightarrow y^\mu(x^\nu)$

(2) Explain why the covariant derivative of the metric tensor is zero.

(3) Consider a satellite in a geo-stationary circular orbit around the Earth. Consider a standard clock at rest at the surface of the Earth and a standard clock at rest in the satellite. The two clocks are being syncronized. How long will it take (measured by the clock at the surface of the Earth) before they differ by one second. (the acceleration at the surface of the Earth is assumed to be $10 \mathrm{m/s}^2$, the radius of the Earth 6000 Km, the radius of a geo-static orbit 36000 Km and the velocity of light 300000 Km/s)

(4) Consider an inertial system with coordinates $X^\mu = (T, X, Y, Z)$. Now make a coordinate transformation to an accelerated coordinate system $x^\nu = (t, x, y, z)$ defined by:

$$X = x \cosh t, \qquad T = x \sinh t, \qquad Y = y, \qquad Z = z.$$

(4a) Write down the proper time line element $d\tau^2$ and the equations of motion for a massive particle in free fall in the coordinate system x^μ.

(4b) Show that the solution to the equations of motion for a particle starting at $x = x_0 > 0$, $y = z = 0$ at time $t = 0$ with velocity zero is

$$x(t) = \frac{x_0}{\cosh t}, \quad y = z = 0.$$

(4c) Find the velocity of the particle and compare with the velocity of light (calculate that too).

(4d) Calculate the proper time it takes to reach $x = 0$ for a freely falling particle starting at $x = x_0$, $y = z = 0$ at time $t = 0$ with velocity zero, measured by a standard clock following freeling falling particle.

(5) Write down Einstein's equations, denoting the gravitational constant G and the cosmological constant Λ.

(6) The Friedmann equation in the case where the cosmological constant is zero is

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho\, a^2.$$

(6a) Write down the Friedmann equation if the cosmological constant Λ is not zero.

Assume the density $\rho = 0$.

6b Solve the Friedmann equation if $\Lambda > 0$ and k takes the three values $0, \pm 1$. In which cases do we have a Big Bang ?