GENERAL RELATIVITY FINAL

01.02.2019, 13:30-16:30

Please write your solutions to each of the four problems on a separate sheet of paper and write your name and student number on each sheet! You have 3 hours. Good luck!

- In this exam we work in units in which $c = 1$. The exam counts as 45% of the grade and contains in total 50 points.

Some formulas you might find useful are the following:

- Connection:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right), \tag{0.1}$$

- Riemann tensor, Ricci tensor and Ricci scalar:

$$R_{\alpha\beta\mu}{}^{\delta} = \partial_\beta \Gamma^{\delta}_{\alpha\mu} - \partial_\alpha \Gamma^{\delta}_{\beta\mu} + \Gamma^{\nu}_{\alpha\mu}\Gamma^{\delta}_{\beta\nu} - \Gamma^{\nu}_{\beta\mu}\Gamma^{\delta}_{\alpha\nu}, \qquad R_{\mu\nu} = R_{\mu\alpha\nu\beta}\, g^{\alpha\beta}, \qquad R = R_{\mu\nu}\, g^{\mu\nu}. \tag{0.2}$$

- Energy-momentum tensor of the electromagnetic field:

$$T_{\mu\nu} = F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \tag{0.3}$$

- Komar mass formula:

$$M = \frac{1}{4\pi G} \int_{\partial\Sigma} d^2x \, \sqrt{\gamma^{(2)}}\, n_\mu \, \sigma_\nu \nabla^\mu K^\nu, \tag{0.4}$$

  where Σ is a spacelike hypersurface and $\gamma^{(2)}$ is the determinant of the metric associated to $\partial\Sigma$, while n_μ and σ_μ are the unit vectors associated to Σ and $\partial\Sigma$ respectively.

- Total electric charge of a spacetime: $Q = -\int_{\partial\Sigma} d^2x \, \sqrt{\gamma^{(2)}}\, n_\mu \, \sigma_\nu F^{\mu\nu}$

■ PROBLEM 1 Theoretical questions. *(10 pts.)*

Answer the following briefly:

(a) *(3 pts.)* Give a coordinate invariant definition of the event horizon. In the particular case of stationary, asymptotically flat black holes with event horizons with spherical topology (such as all the black holes we discussed in class) you can make this explicit by using a radial coordiate r. What is the formula for the event horizon in this case?

(b) *(7 pts.)* Geodesics of the Schwarzschild blackhole satisfy the equation,

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \mathcal{E}\,, \qquad V(r) = \frac{1}{2}\epsilon - \epsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}$$

where $\mathcal{E} = E^2/2$, $\epsilon = +1$ for a massive particle and $\epsilon = 0$ for a massless particle. Here E is the energy of the particle and L is its orbital angular momentum. Which term in this potential distinguishes general relativity from Newtonian gravity? Now, consider two massless particles particle 1 and 2 with energy and angular momentum (E_1, L_1) and (E_2, L_2), respectively, sent from asymptotic infinity to the black hole. You know that $E_1 > E_2$ and $L_1 < L_2$ and that the particle 1 managed to escape the black hole. Will particle 2 be absorbed or reflected?

(c) *Bonus (2 pts.)* What is the definition of a surface of infinite redshift? What is an ergosphere?

(d) *Bonus (2 pts.)* How many degrees of freedom are there in a gravitational wave? What do they correspond to?

■ PROBLEM 2 The extremal Reissner-Nordström solution *(24 points)*

Consider the following static, asymptotically flat metric

$$\begin{aligned} ds^2 &= -H(\rho)^{-2}dt^2 + H(\rho)^2\left[d\rho^2 + \rho^2\,d\theta^2 + \rho^2\sin^2\theta\,d\phi^2\right] \\ &= -H(|\vec{x}|)^{-2}dt^2 + H(|\vec{x}|)^2\left[dx^2 + dy^2 + dz^2\right] \end{aligned} \tag{2.1}$$

together with an electromagnetic potential of the form

$$A_t(\rho) = \frac{H(\rho)^{-1} - 1}{4\pi\sqrt{G}}, \qquad A_r = A_\theta = A_\phi = 0, \tag{2.2}$$

where G denotes Newton's constant.

(a) *(2 pts.)* Name the Killing vectors associated to the metric (2.1). Using the fact that the metric does not depend explicitly on t and ϕ, write down the two conserved quantities along geodesics associated to their Killing vectors.

(b) *(3 pts.)* Compute the components of the energy-momentum tensor and its trace, T^μ_μ. Using this result compute the Ricci scalar R.

(c) *(4 pts.)* Write down Einstein's equations and Maxwell's equations and show that they are simultaneously satisfied by

$$\nabla^2 H(\rho) = 0, \tag{2.3}$$

i.e. $H(\rho)$ satisfies Laplace's equation. The nonzero components of the Ricci tensor are given at the end of this problem.

Hint: Recall that the laplacian in spherical coordinates is given by

$$\nabla^2 f(\rho) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right). \tag{2.4}$$

(d) *(2 pts.)* Write down the most general solution to (2.3). What boundary condition are you going to assume?

From now on we will assume the function $H(\rho)$ has the form

$$H(\rho) = 1 + \frac{C}{\rho}, \tag{2.5}$$

where C is a constant of integration to be determined.

(e) *(4 pts.)* Compute the Komar integrals associated to the mass and charge of the spacetime and use these results to fix the constant C.

(f) *(3 pts.)* Does this spacetime have a curvature singularity and/or an event horizon? If yes, where are they located? Is there a trapped surface?

(g) *(3 pts.)* Using your results from question **(a)** write down the radial geodesic equation for a massless and massive particle moving along the equatorial plane of this spacetime, i.e. $\theta = \frac{\pi}{2}$.

(h) *(3 pts.)* Consider now only the massless case. Find all circular orbits (there are two of them). Are they stable or unstable?

The nonzero components of the Ricci tensor are

$$R_{tt} = \frac{\rho\, H'^2 - H\,(2H' + \rho\, H'')}{\rho\, H^6} \qquad R_{\rho\rho} = -\frac{\rho\, H'^2 + H\,(2H' + \rho\, H'')}{\rho\, H^2}$$

$$R_{\theta\theta} = \frac{\rho^2\, H'^2 - \rho\, H\,(2H' + \rho\, H'')}{H^2} \qquad R_{\phi\phi} = R_{\theta\theta} \sin^2\theta$$

where $H' \equiv \frac{\partial H}{\partial \rho}$.

■ PROBLEM 3 The BTZ black hole *(6 points)*

Consider the following metric in $2+1$ dimensions:

$$ds^2 = -\left(-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}\right) dt^2 + \left(-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}\right)^{-1} dr^2 + r^2 \left(d\phi - \frac{J}{2r^2} dt\right)^2, \tag{3.1}$$

where M and J are interpreted as the mass and the angular momentum of the blackhole. This metric is an exact solution to Einstein's equations in the vacuum with negative cosmological constant $\Lambda = -\ell^{-2}$.

(a) *(3 pts.)* Find the location of the horizons and curvature singularity. Is there an ergosphere?

(b) *(3 pts.)* For what values of the parameters does this black hole become extremal?

■ PROBLEM 4 FLRW spacetime *(10 points)*

The FLRW metric for a flat spatial geometry using a proper time coordinate t is given as,

$$ds^2 = -dt^2 + a^2 \delta_{ij} dx^i dx^j, \qquad i = 1, 2, 3, \tag{4.1}$$

where $a = a(t)$ is a time-dependent scale factor.

(a) *(2 pts.)* Show that the Christoffel connection has components

$$\Gamma^t_{ij} = a\,\dot{a}\,\delta_{ij}, \qquad \Gamma^i_{tj} = \Gamma^i_{jt} = \frac{\dot{a}}{a}\,\delta^i_j, \tag{4.2}$$

while the others may be assumed to vanish.

(b) *(3 pts.)* Making use of its symmetries, show that the only non-vanishing components of the Riemann tensor are

$$R_{tit}{}^{j} = -\frac{\ddot{a}}{a}\,\delta^j_i, \qquad R_{ijk}{}^{l} = \dot{a}^2\left(\delta_{ik}\delta^l_j - \delta_{jk}\delta^l_i\right), \tag{4.3}$$

and that we may express the Ricci tensor components in terms of these as

$$R_{tt} = R_{tit}{}^{i}, \qquad R_{ti} = 0, \qquad R_{ij} = -\,R_{tit}{}^{k}\,g_{kj} + R_{ikj}{}^{k}. \tag{4.4}$$

(c) *(5 pts.)* Write down the Einstein equation for space-time Eq.(4.1) that includes cosmological constant matter with stress tensor $T_{\mu\nu} = -\Lambda g_{\mu\nu}$ for some constant Λ. Solve the tt component of the equation to investigate the evolution of the scale factor $a(t)$. What physically is happening to the universe in this situation?