GENERAL RELATIVITY FINAL

03.02.2023, from 13:30 - 16.30 in EDUC - GAMMA & BBG 2.23 (extra time)

Please write your solutions for each problem on a separate sheet of paper and write your name and student number on each sheet! You have 3 hours. Good luck!

- In this exam we work in units in which $c = 1$. The exam counts as 40% of the grade and contains in total 40 points. In this exam, $G_N = 6.67 \times 10^{-11}\mathrm{m}^3\mathrm{s}^{-2}\mathrm{kg}^{-1}$ denotes Newton's gravitational constant.

Points: 8 (problem 1) + 10 (problem 2) + 5 (problem 3) + 17 + 3 bonus (problem 4)
= 40 + 3 bonus (in total)

■ PROBLEM 1 Theoretical questions. *(8 points)*

Answer the following questions concisely:

(a) *(2 points)* State the Birkhoff theorem.

(b) *(2 points)* What is the set of points where the vector $\partial_\mu r$ is null in black hole spacetimes?

(c) *(2 points)* Which mathematical object can be used to establish that a black hole spacetime possesses a singularity or singularities?

(d) *(2 points)* Describe briefly the experimental setup used by the LIGO/Virgo Collaboration to measure gravitational waves.

■ PROBLEM 2 Palatini formalism. *(10 points)*

The Einstein equation can be derived by varying the action $S = S_{\mathrm{EH}} + S_{\mathrm{matter}}$ with respect to the metric, where the Einstein-Hilbert action is considered a functional of the metric only. That means we have expressed the connection in terms of the metric (Christoffel connection).

The Einstein equation can also be derived by treating the metric and connection as independent degrees of freedom, and varying the action with respect to them separately. This is know as the Palatini formalism, for which the Einstein-Hilbert action can be written as,

$$S_{\mathrm{EH}} = \frac{1}{16\pi G_N}\int \mathrm{d}^4x\sqrt{-g}\, g^{\mu\nu}R_{\mu\nu}(\Gamma)\ , \tag{2.1}$$

where G_N is Newton's constant, $g^{\mu\nu}$ is the inverse metric tensor, $g = \det[g_{\mu\nu}]$ and the Ricci tensor $R_{\mu\nu}$ depends on the (unspecified) connection $\Gamma^{\alpha}_{\ \mu\nu}$ in the usual way,

$$R_{\mu\nu}(\Gamma) = \delta^{\beta}_{\alpha}R^{\alpha}_{\ \mu\beta\nu}(\Gamma) = \delta^{\beta}_{\alpha}\Big(\partial_\beta\Gamma^{\alpha}_{\ \nu\mu} + \Gamma^{\alpha}_{\ \beta\lambda}\Gamma^{\lambda}_{\ \nu\mu} - \partial_\nu\Gamma^{\alpha}_{\ \beta\mu} - \Gamma^{\alpha}_{\ \nu\lambda}\Gamma^{\lambda}_{\ \beta\mu}\Big)\,. \tag{2.2}$$

(a) *(2 points)* Show that the variation with respect to the metric leads to the Einstein equation

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}\ . \tag{2.3}$$

Note that the Einstein tensor $G_{\mu\nu}$ is defined in terms of the Ricci tensor in the usual way, which in turn depends on the yet unspecified connection as in (2.2). In order for this equation to make sense we need to vary the action with respect to the connection.

In the remainder of this problem you will show that, if we assume a torsionless connection, $\Gamma^{\alpha}_{[\mu\nu]} = 0$, the variational principle tells us it also has to be metric compatible, *i.e.* $\nabla_\alpha g^{\mu\nu} = 0$. This then leads to (2.3) being the standard Einstein equation for the metric.

(b) *(1 point)* Write down the variation of the Ricci tensor $\delta R_{\mu\nu}(\Gamma)$ with respect to the connection.

(c) *(2 points)* Utilizing (ordinary) partial integration show that the requirement that the Einstein-Hilbert action be stationary under variation with respect to the connection implies the following relation

$$0 = -\frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}g^{\mu\nu}\right) + g^{\mu\nu}\Gamma^{\sigma}_{\sigma\alpha} - g^{\sigma\nu}\Gamma^{\mu}_{\alpha\sigma} - g^{\mu\sigma}\Gamma^{\nu}_{\sigma\alpha} + \delta^{\nu}_{\alpha}\left[\frac{1}{\sqrt{-g}}\partial_\sigma(\sqrt{-g}g^{\mu\sigma}) + \Gamma^{\mu}_{\sigma\rho}g^{\sigma\rho}\right] . \tag{2.4}$$

Note that we cannot use the covariant version of Stokes' theorem, since it only works for the Christoffel connection, and here the connection is general.

(d) *(2 points)* Recall next the following identity,

$$\frac{1}{\sqrt{-g}}\partial_\alpha\sqrt{-g} = -\frac{1}{2}g_{\rho\lambda}\partial_\alpha g^{\rho\lambda} . \tag{2.5}$$

Using (2.5) and the definition of the covariant derivative, rewrite all the partial derivatives in (2.4) in terms of covariant ones, and show that the condition (2.4) reduces to

$$0 = -\nabla_\alpha g^{\mu\nu} + \frac{1}{2}g^{\mu\nu}g_{\rho\sigma}\nabla_\alpha g^{\rho\sigma} + \delta^{\nu}_{\alpha}\left[\nabla_\lambda g^{\mu\lambda} - \frac{1}{2}g^{\mu\lambda}g_{\rho\sigma}\nabla_\lambda g^{\rho\sigma}\right] . \tag{2.6}$$

(e) *(3 points)* By appropriate contractions of (2.6) with the metric tensor show that (2.6) implies

$$\nabla_\alpha g^{\mu\nu} = 0 , \tag{2.7}$$

that is, the Palatini variational principle implies metric compatibility.
Hint: The appropriate contractions of (2.6) are $g_{\mu\nu}$, and then δ^{α}_{ν}.

■ PROBLEM 3 Fierz-Pauli action. *(5 points)*

In this problem you will construct the gauge-invariant action for a massless spin-2 particle, known as the Fierz-Pauli action.

To begin, consider a slightly perturbed Minkowski space,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) , \tag{3.1}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ is the canonical Minkowski metric, and $h_{\mu\nu}$ is a small perturbation, $|h_{\mu\nu}| \ll 1$ ($\forall\ \mu,\nu$). Under the infinitesimal diffeomorphism transformation (which is an active coordinate transformation, as opposed to a passive one), the metric perturbation transforms as

$$h_{\mu\nu}(x) \to \widetilde{h}_{\mu\nu}(x) = h_{\mu\nu}(x) - \partial_\mu\xi_\nu(x) - \partial_\nu\xi_\mu(x) , \tag{3.2}$$

where $h_{\mu\nu}$ are ξ_μ are of the same order.

One can show that the only Lorentz scalars of the form $\partial h \times \partial h$ are,

$$\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu}\,, \qquad \partial_\rho h_{\mu\nu}\partial^\nu h^{\mu\rho}\,, \qquad \partial_\nu h^{\mu\nu}\partial^\rho h_{\mu\rho}\,, \qquad \partial_\nu h^{\mu\nu}\partial_\mu h\,, \qquad \partial^\mu h \partial_\mu h\,, \tag{3.3}$$

where $h \equiv \eta^{\alpha\beta}h_{\alpha\beta} = h^\alpha_\alpha$ is the trace of the tensor $h_{\alpha\beta}$ and indices are raised/lowered using the Minkowski metric. Based on (3.3), one can show that – up to boundary terms – the quadratic gravitational action is,

$$S = \int \mathrm{d}^4x \Big(a_1 \partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + a_2 \partial_\rho h_{\mu\nu}\partial^\nu h^{\mu\rho} + a_3 \partial_\nu h^{\mu\nu}\partial_\mu h + a_4 \partial^\mu h \partial_\mu h \Big)\,, \tag{3.4}$$

where a_1, a_2, a_3 and a_4 are constant real numbers (coupling constants).

Now we want to impose the gauge symmetry of linearized gravity, namely the condition that the action (3.4) has to be invariant under infinitesimal transformations of the form (3.2).

(a) *(4 points)* Show that imposing gauge invariance of the action (3.4) constrains all the coefficients a_i except for the overal normalization a_1, reducing the action to,

$$S_{FP} = a_1 \int \mathrm{d}^4x \left(\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - 2\partial_\rho h_{\mu\nu}\partial^\nu h^{\mu\rho} + 2\partial_\nu h^{\mu\nu}\partial_\mu h - \partial^\mu h \partial_\mu h \right)\,. \tag{3.5}$$

The gauge-invariant action (3.5) is known as the Fierz-Pauli action (1939).

Hint: Using integration by parts when necessary, show that under (3.2), the action (3.4) transforms as $S \to \tilde{S}$ with

$$\begin{aligned}\tilde{S} = S \; &- \int d^4x \left[(4a_1 + 2a_2)\partial_\rho h_{\mu\nu}\partial^\rho\partial^\mu\xi^\nu + (2a_2 + 2a_3)\partial_\rho h_{\mu\nu}\partial^\mu\partial^\nu\xi^\rho + (2a_3 + 4a_4)\partial^\mu h \partial_\mu \partial_\nu \xi^\nu \right] \\ &+ \int d^4x \left[(2a_1 + a_2)\partial^\rho\partial^\mu\xi^\nu\partial_\rho\left(\partial_\mu\xi_\nu + \partial_\nu\xi_\mu\right) + \left(2a_2 + 4a_3 + 4a_4\right)\partial_\mu\partial_\nu\xi^\nu\partial^\mu\partial_\rho\xi^\rho \right]\,. \end{aligned} \tag{3.6}$$

Then observe that the conditions $a_2 = -2a_1$, $a_3 = 2a_1$ and $a_4 = -a_1$, which make the first integral of Eq. (3.6) vanish, also give zero for the second integral of Eq. (3.6).

(b) *(1 point)* Show that in the traceless-transverse gauge, in which $h_{0\mu} = 0$, $h_{ij} \to h^{TT}_{ij}$, with $\partial_i h^{TT}_{ij} = 0$ and $\delta_{ij}h^{TT}_{ij} = 0$, the action (3.5) reduces to,

$$S_{FP} = a_1 \int \mathrm{d}^4x \Big(\partial_\rho h^{TT}_{ij} \partial^\rho h^{TT}_{ij} \Big)\,, \tag{3.7}$$

which can be used to derive the vacuum wave equation for gravitational waves.

■ PROBLEM 4 Light deflection in a black hole spacetime. *(17 points + 3 bonus points)*

The Schwarzschild line element (in units $c = 1$) is,

$$\mathrm{d}s^2 = -\left(1 - \frac{r_S}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{r_S}{r}} + r^2 \mathrm{d}\Omega^2\,, \tag{4.1}$$

where the Schwarzschild radius r_S is given in terms of the mass M of the gravitating object as $r_S = 2G_N M$, and we have defined $\mathrm{d}\Omega^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2$.

(a) *(3 points)* Consider a redefinition of the radial coordinate

$$r = \bar{r}\left(1 + \frac{r_S}{4\bar{r}}\right)^2 , \tag{4.2}$$

which is made bijective by mapping the interval $r > r_S$ to $\bar{r} > r_S/4$. Show that, in terms of the coordinates $(t, \bar{r}, \vartheta, \varphi)$, the line element takes the form

$$\mathrm{d}s^2 = -\left(\frac{1 - \frac{r_S}{4\bar{r}}}{1 + \frac{r_S}{4\bar{r}}}\right)^2 \mathrm{d}t^2 + \left(1 + \frac{r_S}{4\bar{r}}\right)^4 \left(d\bar{r}^2 + \bar{r}^2 \mathrm{d}\Omega^2\right) . \tag{4.3}$$

The new coordinate system is called *isotropic coordinates*. Can you justify this name?
Hint: Notice that the spatial part of the metric (4.3) is proportional to $\mathrm{d}\bar{r}^2 + \bar{r}^2\mathrm{d}\Omega^2$.

(b) *(2 points)* Show that, far from the source where $\bar{r} \gg r_S$, the metric can be approximated at leading order in $r_S/\bar{r}$ by

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\left(1 - \frac{r_S}{\bar{r}}\right)\mathrm{d}t^2 + \left(1 + \frac{r_S}{\bar{r}}\right)\left(\mathrm{d}\bar{r}^2 + \bar{r}^2\mathrm{d}\Omega^2\right) . \tag{4.4}$$

Does this result make sense in view of the Newtonian limit?
Hint: Recall the asymptotic form of the metric in terms of the gravitational potential.

In the following, you will study deflection of light in the approximate metric (4.4), by analyzing null geodesics $x^\mu(\lambda)$ parametrized by an affine parameter λ.

As usual, symmetries of the metric (4.4) play an important role because they provide conserved quantities along the geodesics. First, from two of the Killing vectors associated to spherical symmetry, the motion takes place on a plane. Without loss of generality, this can be taken to be the equatorial plane

$$\vartheta = \frac{\pi}{2} . \tag{4.5}$$

(c) *(3 points)*

- Compute the conserved magnitude of the angular momentum $L = R_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$ where $R = \partial_\varphi$ is the remaining rotational Killing vector (in components $R^\mu = (0, 0, 0, 1)$).
- Compute also the conserved energy $E = -K_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$ where $K = \partial_t$ is the Killing vector associated to invariance of the metric (4.4) under time translations (in components $K^\mu = (1, 0, 0, 0)$).
- Use these relations to obtain $\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda}$ and $\frac{\mathrm{d}t}{\mathrm{d}\lambda}$ as functions of $\bar{r}$. Assume that the condition $r_S \ll \bar{r}$ is respected on the full path of light and work at leading order in $r_S/\bar{r}$. You should get, within this approximation,

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\lambda} = \frac{L}{\bar{r}^2}\left(1 - \frac{r_S}{\bar{r}}\right) . \tag{4.6}$$

$$\frac{\mathrm{d}t}{\mathrm{d}\lambda} = E\left(1 + \frac{r_S}{\bar{r}}\right) , \tag{4.7}$$

(d) *(2 points)* Now impose the null trajectory condition, which must be respected by light, in the metric (4.4),

$$g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0 , \tag{4.8}$$

and substitute equations (4.7) and (4.6) to get (again at leading order in $r_S/\bar{r}$)

$$\left(\frac{\mathrm{d}\bar{r}}{\mathrm{d}\lambda}\right)^2 = E^2 - \frac{L^2}{\bar{r}^2}\left(1 - 2\frac{r_S}{\bar{r}}\right) . \tag{4.9}$$

(e) *(1 point)* Combine equations (4.9) and (4.6) to obtain the following approximate equation for the trajectory:

$$\left(\frac{d\bar{r}}{d\varphi}\right)^2 = \frac{\bar{r}^4}{b^2}\left(1 + 2\frac{r_S}{\bar{r}}\right) - \bar{r}^2, \tag{4.10}$$

where $b \equiv L/E$ is called the *impact parameter.*

Consider a beam of light with $L > 0$ (and $E > 0$) incoming from large distance $\bar{r} \to \infty$ and choose $\varphi = 0$ at that faraway initial condition.

(f) *(3 points)* Before solving Eq. (4.10) it is useful to understand the meaning of the impact parameter b. First assume that there is no source mass at the origin of the coordinate system, *i.e.* assume that $r_S = 0$. Then we expect light to follow a straight path. Check that, when $r_S = 0$, the equation (4.10) is solved by $\bar{r} \sin\varphi = b$. Argue that this is indeed the equation of a straight line and that the impact parameter b is the minimum distance from the origin reached. Sketch this straight line and explain the meaning of the impact parameter.

(g) *(3 points)* Now consider $r_S \neq 0$. It is convenient to trade $\bar{r}$ for the new variable $u = r_S/\bar{r}$. Show that Eq. (4.10) is equivalent to,

$$\left(\frac{du}{d\varphi}\right)^2 = \left(\frac{r_S}{b}\right)^2 (1 + 2u) - u^2. \tag{4.11}$$

Show that the minimum coordinate distance $\bar{r}_{\text{min}}$ from the source reached in this case is

$$\bar{r}_{\text{min}} = \sqrt{b^2 + r_S^2} - r_S. \tag{4.12}$$

This is consistent with the assumed approximation that for all points of the trajectory $\bar{r} \gg r_S$, provided that $b \gg r_S$. Why?

The variable $\bar{r}$ starts from large values ($\bar{r} \to \infty$), decreases down to its minimum value $\bar{r}_{\text{min}}$ and then grows again reaching asymptotically a new straight line. Clearly the trajectory is symmetric with respect to the straight line connecting the source M (the origin) to the point of minimum distance, so it is sufficient to study the first half of the trajectory, namely when the light beam coming from infinity approaches the mass source until reaching a minimum distance.

(h) *(3 bonus points)* Integrate Eq. (4.11) in the first half of the travel to show that, at the minimum distance, φ is given by

$$\varphi(\bar{r}_{\text{min}}) = \frac{\pi}{2} + \arcsin\left(\frac{r_S}{\sqrt{b^2 + r_S^2}}\right) \simeq \frac{\pi}{2} + \frac{r_S}{b}, \tag{4.13}$$

where in the last step we used the consistency requirement $r_S \ll b$ found in Question **(g)**. Argue, by sketching a figure, that the total deflection angle of light, including the second half of the travel, is then

$$\Delta\varphi = 2\varphi(\bar{r}_{\text{min}}) - \pi \simeq \frac{2r_S}{b} = \frac{4G_N M}{b}. \tag{4.14}$$

Hint: Pay attention to choosing the right sign in front of the integral and remember the initial condition $\varphi = 0$ at $\bar{r} \to \infty$. The following indefinite integral is needed, with $\alpha = r_S/b$,

$$\int \frac{du}{\sqrt{\alpha^2(1 + 2u) - u^2}} = \arcsin\left(\frac{u - \alpha^2}{\alpha\sqrt{1 + \alpha^2}}\right). \tag{4.15}$$