Department of Physics and Astronomy, Faculty of Science, UU.
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In 2005/2006, the course NS-TP428M was given by Renate Loll.

General Relativity (NS-TP428M)
30 January 2006

Question 1: A well-known cosmological solution *(7 points)*

Consider the Robertson-Walker metric in standard comoving coordinates for the special case of negative curvature (k= -1) and for a scale factor $a(t) = t$

a) By making a suitable transformation to new coordinates, show that this universe is simply Minkowski space in disguise. (Hint: keep the angular variables in the two-dimensional spherical volume element $d\Omega^2$ fixed, so that you end up with Minkowski space in radial coordinates.) *(6 points)*

b) Determine the domains in terms of the original coordinates (t, r, θ, ϕ) of the Robertson-Walker metric where the coordinate transformation of (a) is well-defined. *(1 point)*

Question 2: Parallel-transporting *(13 points)*

Consider the two-sphere with metric inherited from $\mathbb{R}^3$ and unit radius $r = 1$, and parametrized in terms of standard coordinates (θ, ϕ). (Hints: part (c) and (d) below are largely independent of (a) and (b). Try and do at least part of (b).)

a) Making use of the non-vanishing Christoffer symbols $\Gamma^{\phi}_{\phi\theta} = \frac{\cos\theta}{\sin\theta}$, $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$, write down in as explicit a form as possible the equations for parallel-transport on the sphere of a vector V^μ along a curve $\gamma(t) = (\theta(t), \phi(t))$ with tangent vector $t^\mu(t)$. *(2 points)*

b) Start with the vector V^μ which is the unit vector in θ-direction. How does this vector behave when it is parallel-transported along the closed curve

$$\gamma = \gamma_4 \circ \gamma_3 \circ \gamma_2 \circ \gamma_1 \tag{1}$$

consisting of the four pieces

$$\begin{array}{rcll} \gamma_1(t) & = & (\frac{\pi}{2}, t) & \text{for } 0 \le t \le t_1, \\ \gamma_2(t) & = & (\frac{\pi}{2} - t, t_1) & \text{for } 0 \le t \le t_2, \\ \gamma_3(t) & = & (\frac{\pi}{2} - t_2, t_1 - t) & \text{for } 0 \le t \le t_1, \\ \gamma_4(t) & = & (\frac{\pi}{2} - t_2 + t, 0) & \text{for } 0 \le t \le t_2, \end{array}$$

with $0 \le t_1 \le 2\pi$, $0 \le t_2 \le \frac{\pi}{2}$? (By definition, a curve $\xi \circ \eta$ is the curve given by first moving along η and then along ξ.) It may be helpful to start by making a sketch of the geometry of the problem. *(7.5 points)*

c) Compute the Ricci tensor an the Ricci scalar of the spherical surface. *(2 points)*

d) Show by explicit calculation that the angle $\Delta\rho$ by which the vector V^μ has been rotated after parallel-transport around the curve (1) equals

$$\Delta\rho = \frac{1}{2}\int_A R, \tag{2}$$

where A is the surface enclosed by the curve γ, and R is the Ricci scalar. *(1.5 points)*