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In 2007/2008, the course NS-TP428m was given by Renate Loll.

General Relativity (NS-TP428m)
March 3, 2008

Question 1: A black hole metric *(20 points)*

In the coordinates $x^\mu = (T, r, \theta, \phi)$, consider the black-hole metric

$$ds^2 = -dT^2 + (dr + \sqrt{\frac{c}{r}} dT)^2 + r^2 d\Omega^2_{(2)}, \tag{1}$$

where $c > 0$ is a constant, and $d\Omega^2_{(2)} = d\theta^2 + \sin^2\theta\, d\phi^2$ is the standard metric on the unit two-sphere. [Note that various of the subproblems below are computationally independent, so read everything through before starting to calculate.]

a) For which range of coordinates is the metric (1) well defined and nonsingular? lf applicable, can you comment on the nature of the singularities? (You are not expected to calculate curvature tensors here!) *(1 point)*

b) Which value of r corresponds to an event horizon? Prove your assertion by referring to geometric properties directly extracted from expression (1) (without using any other coordinate system, for example). – Recall that an "event horizon" was introduced in the lectures as a particular hypersurface from beyond which "not even light rays can escape". *(3 points)*

c) Identify conserved quantities of (massive) particles moving along geodesics of the metric *(2 points)*

d) In the spacetime given by (1), an otherwise freely moving (massive) particle is emitted from a radial coordinate r_0 which lies outside the event horizon, with initial proper angular velocity $d\phi/d\tau = \omega$ (in the plane $\theta = \pi/2$), and perpendicularly to the radial direction. (lt may be helpful to make a sketch of the spatial geometry of the situation.). Find the minimum value of ω for which the particle can overcome the gravitational attraction of the black hole, that is, for which it reaches (spatial) infinity, as function of r_0 and c. *(4 points)*

e) Try to find a coordinate transformation from (T, r, θ, ϕ) to Schwarzschild coordinates (t, r, θ, ϕ) which transforms the metric (1) to the Schwarzschild solution in standard form. (For definiteness, consider the region outside the event horizon.) Does such a transformation exist? If not, what is the problem? If yes, what is the transformation? (Hint: $\int dx \frac{\sqrt{x}}{x-1} = 2\sqrt{x} + \log \frac{\sqrt{x}-1}{\sqrt{x}+1}$.) *(3 points)*

f) Consider the three-dimensional hypersurfaces of constant time (T and t respectively) both for the metric (1) and the Schwarzschild solution in standard Schwarzschild coordinates. Determine the two metrics induced on these hypersurfaces from the full spacetime metrics by setting $T = const.$ and $t = const.$ respectively. Compute the associated *three*-dimensional Ricci and scalar curvatures for either case. Do these metrics satisfy three- dimensional vacuum "Einstein equations"? Is your result compatible with the outcome of e) above? (Hint: you may use that the off-diagonal elements of both Ricci tensors vanish.) *(5 points)*

g) Radial curves in the spacetime (1) with $dr = -\sqrt{c/r}\; dT$ are timelike, and T is the proper time along these curves. Show that they are orthogonal (in the sense of the four-dimensional spacetime metric) to the surfaces of constant T. *(2 points)*