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In 2008/2009, the course NS-TP526M was given by Dr.S.Vandoren.

String Theory (NS-TP526M)
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Question 1. Scale (Weyl) invariance of the superstring

The action for the superstring is given by

$$S = -\frac{T}{2}\int d^2\sigma e\left[h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu + 2i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu - i\bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi^\mu (\partial_\beta X_\mu - \frac{i}{4}\bar{\chi}_\beta \psi_\mu)\right], \tag{1}$$

where $h_{\alpha\beta} = e^a_\alpha \eta_{ab} e^b_\beta$, with $e \equiv det(e^a_\alpha)$, $\eta_{ab} = diag(-1,1)$ and $h^{\alpha\beta}$ the inverse worldsheet metric. Furthermore $\rho^\alpha = e^\alpha_a \rho^a$, with e^α_a the inverse zweibein and ρ^a are the Dirac matrices in two dimensions,

$$\rho^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

The Majorana fermions ψ^μ and χ_α are spinors with two real components and $\bar{\psi} \equiv \psi^\dagger \rho^0$, and similarly for χ.

a) Show that for any two Majorana spinors χ and ψ,

$$\bar{\chi}\rho^a \psi = -\bar{\psi}\rho^a \chi. \tag{3}$$

b) Show that the superstring acrion is invariant under local Weyl rescalings of the fields

$$X^\mu \to X^\mu, \quad e^a_\alpha \to \Lambda e^a_\alpha, \quad \psi^\mu \to \Lambda^{-\frac{1}{2}}\psi^\mu, \quad \chi_\alpha \to \Lambda^{\frac{1}{2}}\chi_\alpha, \tag{4}$$

for any function of the worldsheet coordinates $\Lambda(\sigma,\tau)$.

Question 2. Worldsheet Hamiltonian: $H = L_0$.

In light-cone coordinates $\sigma^\pm = \tau \pm \sigma$, the action for the superstring in superconformal gauge is

$$S = 2T\int d^2\sigma \left[\partial_+ X^\mu \partial_- X_\mu + i(\psi^\mu_+ \partial_- \psi_{+,\mu} + \psi^\mu_- \partial_+ \psi_{-,\mu})\right], \tag{5}$$

where $\psi_\pm$ are the two real components of the Majorana spinor ψ, and $\partial_\pm = \partial/\partial\sigma^\pm$. We can write the action as $S = \int d^2\sigma \mathcal{L}$ with Lagrangian density $\mathcal{L}$. The Hamiltonian density then follows from the general formula $\mathcal{H} = p\dot{q} - \mathcal{L}$, where the momentum p conjugate to q is given by $p \equiv \partial L/\partial \dot{q}$. The Hamiltonian H is then given by the spatial integral of $\mathcal{H}$.

a) Show that fot the superstring the Hamiltonian density is given by

$$\mathcal{H} = \frac{T}{2}\left[\left(\dot{X}^\mu\right)^2 + (X'^\mu)^2\right] + iT\left[\psi^\mu_+ \psi'_{+,\mu} - \psi^\mu_- \psi'_{-,\mu}\right], \tag{6}$$

where $\dot{}$ denotest the time derivative and $'$ denotes the derivative with respect to σ.

For the open superstring, the Hamiltonian is

$$H = \int_0^\pi d\sigma \mathcal{H}, \tag{7}$$

andt the mode expansions of the fields are

$$
\begin{aligned}
X^\mu(\tau,\sigma) &= x^\mu + \frac{p^\mu}{\pi T}\tau + \frac{i}{\sqrt{\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau} cos(n\sigma), &(8)\\
\psi_\pm^\mu(\tau,\sigma) &= \frac{1}{2\sqrt{\pi T}}\sum_{r\in Z+\theta} b_r^\mu e^{-ir\sigma^\pm}, &(9)
\end{aligned}
$$

where r runs over the integers in the Ramond sector $\theta = 0$ and over the half integers $\theta = \frac{1}{2}$ in the Neveu-Schwarz sector.

b) In terms of the oscillator modes, show that the Hamiltonian becomes

$$
H = \tfrac{1}{2}\left(\sum_{n=-\infty}^{\infty}\alpha_n^\mu\alpha_{-n,\mu} - \sum_{r\in Z=\theta} r b_r^\mu b_{-r,\mu}\right), \tag{10}
$$

with $\alpha_0^\mu = \frac{1}{\sqrt{\pi T}}p^\mu$. (This expression is in fact the same as one of the Virasoro generators, i.e. $H = L_0$)

Question Open string states

Consider the open superstring, quantized in the light-cone gauge, in which the only independent oscillator modes are transversal and denoted by α_n^i and b_r^i where $u = 1,\ldots,D-2 = 8$. In the quantum theory, one imposes the commutation relations $[\alpha_m^i, \alpha_n^j] = m\delta_{m+n,0}\delta^{i,j}$, andt anti-commutation relations $b_r^i, b_s^j = \delta^{ij}\delta_{r+s}$.
The mass operator, in dimensionless units, is given by

$$
M^2 = \sum_{m=1}^{\infty}\alpha_{-m}^i\alpha_m^i + \sum_{r\in Z+\theta} r b_{-r}^i b_r^i - a, \tag{11}
$$

where the normal ordering constant $a = \frac{1}{2}$ in the Neveu-Schwarz (NS) sector, and $a = 0$ in the Ramond (R) sector.

a) Construct the states in the NS sector of the open superstring spectrum, at mass level $M^2 = 1$. [Hint: there are three types of states]

b) Construct the states in the R sector of the open superstring spectrum, also at mass level $M^2 = 1$.

c) Which states are projected out by the GSO projection? Show that after the GSO projection, one ends up with an equal number of bosonic and fermionic states.
(To remind you, the GSO operator in the NS sector is defined by $G_{NS} = -(-)^{N_b}$, where $N_b = \sum_r b_{-r}^i b_r^i$. Similarly, for the R sector, we have $G_R = \Gamma^9(-)^{N_b}$, where $\Gamma^9 = \pm 1$ measures the chirality of spinors in eight dimensions.)