DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, UU.
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Solutions[1] Deeltentamen A Wat is Wiskunde? (WISB101) 2 november 2009

Question 1

Calculating the truth tables of both expressions one sees that the two expressions take on the same truth value for each combination of truth values for P, Q, R. Thus they are logically equivalent. We leave the details of calculating the truth values to the reader (of course, this calculation should not be neglected in a full answer!).

Question 2

We prove by induction on n that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$.
For $n = 1$ the left hand side becomes 2 while the right hand side is $\frac{1 \cdot 2 \cdot 3}{3} = 2$ thus establishing the induction base. Assume now that the equality holds for a given natural number k and we set out to prove that it also holds for $k+1$. Then our induction hypothesis is that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}$$

and we wish to prove that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}.$$

We calculate the left hand side of the last equality:

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (k+1)(k+2) = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1) + (k+1)(k+2).$$

Here we can use the induction hypothesis to replace the sum of the first k summands on the right hand side by $\frac{k(k+1)(k+2)}{3}$, thus we conclude that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (k+1)(k+2) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2).$$

But

$$\frac{k(k+1)(k+2)}{3} + (k+1)(k+2) = \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} = \frac{(k+3)(k+1)(k+2)}{3}$$

precisely as required to establish the induction step. We conclude, by the principle of mathematical induction, that the general formula holds for all natural numbers n.

Question 3

a) Let $x \in (A-B) \cap (A-C)$. We would like to show that $x \in A-(B \cup C)$. Since $x \in (A-B) \cap (A-C)$ it follows that $x \in A - B$ and $x \in A - C$. Which means that $x \in A$ and $x \notin B$ and $x \notin C$. Since $x \notin B$ and $x \notin C$ it follows that $x \notin B \cup C$. Together with $x \in A$ we conclude that $x \in A - (B \cup C)$. We thus have proved that $(A - B) \cap (A - C) \subseteq A - (B \cup C)$. Now let $y \in A - (B \cup C)$. Then $y \in A$ and $y \notin B \cup C$. Since $y \notin B \cup C$ it follows that $y \notin B$ and $y \notin C$. Since $y \in A$ we conclude that $y \in A - B$ and $y \in A - C$ which means that $y \in (A - B) \cap (A - C)$. Thus we established that $A - (B \cup C) \subseteq (A - B) \cap (A - C)$ which together with $(A - B) \cap (A - C) \subseteq A - (B \cup C)$ proves that $(A - B) \cap (A - C) = A - (B \cup C)$ as desired.

[1]These solutions were made with great precaution. In case of errors, the TBC cannot be held responsible. However, she will be glad to be informed: `tbc@a-eskwadraat.nl`

b) We provide a counter-example to show that $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$ does not hold in general. Let

$$\begin{aligned} A &= \{1\} \\ B &= \{2\} \\ C &= \{3\} \\ D &= \{4\}. \end{aligned}$$

Then $(A \times B) \cup (C \times D) = \{(1,2)\} \cup \{(3,4)\} = \{(1,2),(3,4)\}$ which contains two elements. On the other hand $(A \cup C) \times (B \cup D) = \{1,3\} \times \{2,4\} = \{(1,2),(1,4),(3,2),(3,4)\}$ which contains four elements. Thus these two sets are clearly not identical and so our counter-example is sufficient.

Question 4

a) We show that the given relation R is not transitive. For that we must find integers a, b, c so that aRb and bRc hold but aRc does not hold. Let $a = 1$, $b = 2$, and $c = 4$. Then aRb holds since $a + b = 3$ which is clearly divisible by 3. Likewise, bRc holds since $b + c = 6$ which is divisible by 2 (also by 3 but this is not important). However, aRc does not hold since $a + c = 5$ which is not divisible by neither 2 nor 3. Thus R is indeed not transitive and thus not an equivalence relation.

b) To prove that S is an equivalence relations we prove that it is reflexive, symmetric, and transitive. To show reflexivity let x be a real number. xSx holds precisely when $x^2 = x^2$, which is clearly the case. Thus for all $x \in \mathbb{R}$ we have xSx which means S is transitive. To show S symmetric let x, y be two real numbers and assume xSy holds. Thus $x^2 = y^2$ which of course implies $y^2 = x^2$ which means that ySx. This establishes symmetry. To establish transitivity of S let x, y, z be real numbers and assume xSy and ySz. This means that $x^2 = y^2$ and that $y^2 = z^2$. This clearly implies $x^2 = z^2$ which means xSz, and thus that S is transitive.

c) We now determine the equivalence class $[a]$ of an arbitrary real number a. We use the definition of an equivalence class:

$$[a] = \{x \in \mathbb{R} \mid aSx\} = \{x \in \mathbb{R} \mid a^2 = x^2\} = \{a, -a\}.$$

Thus, as long as $a \neq -a$ we see that each equivalence class has precisely two elements. It holds that $a = -a$ only for the number $a = 0$, in which case $[0] = \{0\}$ has just one element. For all other $a \neq 0$ the equivalence class $[a]$ has two elements.

Question 5

a) This is not true. For a counter-example see Problem D2. There an equivalence relation S on $\mathbb{R}$ is given such that each equivalence class contains one or two elements while the entire set $\mathbb{R}$ contains infinitly many elements.

b) This is true. We use the fact that the product of two rational numbers is rational, which we first prove. Let x, y be two rational numbers. Then they can be written as $x = \frac{p}{q}$ and $y = \frac{r}{s}$, where $p, q, r, s \in \mathbb{Z}$ and $q \neq 0$ and $s \neq 0$. Now we have

$$xy = \frac{p}{q}\frac{r}{s} = \frac{pr}{qs}$$

and this is again a rational number, as desired. Now to prove the result we prove the contrapositive, namely: if x, y, z are all rational then $x \cdot y \cdot z$ is a rational number. Since x and y are rational it follows that $x \cdot y$ is rational. Since $(x \cdot y)$ and z are rational it follows that $(x \cdot y) \cdot z$ is rational, as desired.

c) This is true. We use the fact that the product of a non-zero rational number by an irrational number is irrational. Note that

$$\sqrt{600} = \sqrt{100 \cdot 6} = \sqrt{100} \cdot \sqrt{6} = 10 \cdot \sqrt{6}.$$

In one of the exercises it was proved that $\sqrt{6}$ is irrational. Since 10 is rational it follows from the result stated above that $\sqrt{600}$ is irrational.