Problem A)

(1) Prove that $(P \Rightarrow (Q \Rightarrow R)) \Leftrightarrow ((R \Rightarrow Q) \Rightarrow P)$ and $P \wedge ((\sim Q) \vee R) \vee (Q \wedge (\sim R))$ are logically equivalent.

(2) Prove that the following is not true: For any statement holds that it is either a tautology or a contradiction.

Problem B) In this problem A, B, C, D are arbitrary sets.

(1) Prove that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

(2) Show that the equality $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$ does not neccesarily hold.

Problem C)

(1) Prove by induction that for every natural number $n > 0$ holds $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$.

Problem D) In each of the following a set A is given together with a relation R on it. In each case state (with proof) whether the relation is an equivalence relation or not. For each case in which R is an equivalence relation determine the equivalence classes.

(1) $A = \mathbb{Z}$ and for $x, y \in A$ holds xRy exactly when 2 or 3 divide $x + y$.

(2) $A = \mathbb{R}$ and for $x, y \in A$ holds xRy exactly when $x^2 = y^2$.

Problem E) For each of the following statements decide if it is true or false. Give a short argument to support your answer.

(1) If R is an equivalence relation on a set A and for every $a \in A$ the equivalence class $[a]$ contains only finitely many elements, then the set A must be finite.

(2) If x, y, z are real numbers such that $x \cdot y \cdot z$ is irrational then at least one of the numbers x, y, z must be irrational.

(3) Prove that $(P \Rightarrow (Q \wedge S) \vee (R \Rightarrow Q)) \wedge (Q \wedge \sim Q) \wedge (((\sim R) \Rightarrow (\sim Q)) \vee R \vee S)$ is a contradiction.

(4) The number $\sqrt{666}$ is irrational.