SOLUTIONS

Problem A:

(1) The function is only not defined when the denominator is 0. This happens only if $x = 2$, thus the requested set A is the set $\mathbb{R} - \{2\}$.

(2) To find the image of f we look at the equation $f(x) = y$ where y is given and solve for x:

$$\frac{2x}{x-2} = y \implies 2x = (x-2)y = xy - 2y \implies (2-y)x = -2y.$$

If $y \neq 2$ then we obtain $x = \frac{2y}{y-2}$ as a solution. Thus any $y \neq 2$ in $\mathbb{R}$ is in the image of f. Now, the equation

$$\frac{2x}{x-2} = 2$$

gives

$$2x = 2x - 4$$

which clearly has no solution. Thus we see that $y = 2$ is not in the image of f. We conclude that $B = \mathbb{R} - \{2\}$.

(3) The inverse of $f : A \to B$ is the function $g : B \to A$ given by $g(y) = \frac{2y}{y-2}$. We verify that by computing

$$f \circ g(y) = \frac{2g(y)}{g(y) - 2} = \frac{2\frac{2y}{y-2}}{\frac{2y}{y-2} - 2} = \frac{\frac{4y}{y-2}}{\frac{4}{y-2}} = y,$$

for all $y \in B$. Since $f = g$ as functions it also follows that $g \circ f(x) = x$ for all $x \in A$. This proves that g is the inverse of f and thus also that f is bijective.

Problem B:

(1) The Schroeder-Bernstein Theorem states that given two sets A and B and injections $f : A \to B$ and $g : B \to A$ then $|A| = |B|$,that is there exists a bijection $h : A \to B$.

(2) To prove $|[-1,1]| = |(-1,1)|$ it suffices, according to the Schroeder-Bernstein Theorem to find injective functions $f : [-1,1] \to (-1,1)$ and $g : (-1,1) \to [-1,1]$. An obvious choice for g is the function given by $g(x) = x$ for all $x \in (0,1)$ which is injective since for any $x, y \in (-1,1)$ if $g(x) = g(y)$ then $x = y$ by definition of g. To find the injection f we need to contract $[-1,1]$ to fit in $(-1,1)$. We can take, for example, the function $f(x) = \frac{x}{2}$ for all $x \in [-1,1]$. We need to verfiy that the codomain of f is indeed $(-1,1)$. This is true since for any $x \in [-1,1]$ holds that $|f(x)| < \frac{1}{2}$, thus $f(x) \in (-\frac{1}{2}, \frac{1}{2}) \subseteq (-1,1)$. Moreover, f is injective since for any $x, y \in [-1,1]$ if $g(x) = g(y)$ then $\frac{x}{2} = \frac{y}{2}$ which implies $x = y$.

(3) There are several ways to prove the desired result. One uses the fact, proved in the lectures, that the set $\mathbb{Q}$ or rational numbers is countable and the theorem that an infinite subset of a countable set is countable. Now, any real number with a finite decimal expansion is rational and thus $T \subseteq \mathbb{Q}$. We thus only need show that T is infinite. Indeed for any natural number n let $x_n = 0.111 \cdots 1$ with 1 repeated n times. These are all distinct numbers and all belong to T thus T is infinite and countable.

Problem C:

(1) We use the Euclidean algorithm by repeatedly applying the division algorithm:

$$\begin{aligned} 1005 &= 10 \cdot 99 + 15 \\ 99 &= 6 \cdot 15 + 9 \\ 15 &= 1 \cdot 9 + 6 \\ 9 &= 1 \cdot 6 + 3 \\ 6 &= 2 \cdot 3 + 0 \end{aligned}$$

and thus $gcd(1005, 99) = 3$. Working backwords from these equations we obtain:

$$\begin{aligned} 3 &= 1 \cdot 9 - 1 \cdot 6 \\ &= 2 \cdot 9 - 1 \cdot 15 \\ &= 2 \cdot 99 - 13 \cdot 15 \\ &= 132 \cdot 99 - 13 \cdot 1005 \end{aligned}$$

thus $x = -13$ and $y = 132$, as desired.

(2) We use the following two properties of $gcd(a, b)$. First that there are always $x, y \in \mathbb{Z}$ such that $xa + yb = gcd(a, b)$. Second, that if there are numbers $u, v \in \mathbb{Z}$ such that $ua + vb = 1$ then $gcd(a, b) = 1$. Now to prove the result: Since $gcd(a, b) = 1$ there exist $x, y \in \mathbb{Z}$ for which

$$xa + yb = 1.$$

Raising this equality to its second power we obtain

$$x^2a^2 + 2xyab + y^2b^2 = 1.$$

Rearranging we get

$$x^2a^2 + (2xya + y^2b)b = 1,$$

and so if we denote $u = x^2$ and $v = 2xya + y^2b$, and noting that both these numbers are integers, we get $ua^2 + vb = 1$ which implies $gcd(a^2, b) = 1$.

Problem D:

(1) Let $\psi, \varphi : (\mathbb{Z}_4, +) \to (\mathbb{Z}_5^*, \cdot)$ be two isomorphisms with $\psi([1]) = \varphi([1])$. Then using the definition of isomorphism we have:

$$\psi([2]) = \psi([1] + [1]) = \psi([1]) \cdot \psi([1]) = \varphi([1]) \cdot \varphi([1]) = \varphi([2])$$

$$\psi([3]) = \psi([2]+[1]) = \psi([2])\cdot\psi([1]) = \varphi([2])\cdot\varphi([1]) = \varphi([3])$$
$$\psi([0]) = \psi([3]+[1]) = \psi([3])\cdot\psi([1]) = \varphi([3])\cdot\varphi([1]) = \varphi([0])$$

thus we see that ψ and φ are identical functions, as needed.

(2) According to part 1 of this problem any isomorphism ψ is completely determined by the element $\psi([1])$. Thus there are at most 4 possible isomorphisms $\psi_1, \psi_2, \psi_3, \psi_4$ with $\psi_i([1]) = [i]$ for $i = 1, 2, 3, 4$. Since any isomorpihsm maps the identity element to the identity element we must have $\psi_1([0]) = [1] = \psi_1([1])$. Thus ψ_1 is not bijective and thus we are left with three possibilites: ψ_2, ψ_3, ψ_4.

(3) To find all subgroups of $(\mathbb{Z}_4, +)$ we consider the relevant subsets of $\mathbb{Z}_4$ and check them with the subgroup test. Since any subgroup must contain the identity element of the group we need only consider those subsets of $\mathbb{Z}_4$ that contain [0]. There are 8 such subsets and one can subject each of them to the subgroup test. One can filter some more sets by using Lagrange's Theorem that says that the order of a subgroup must devide the order of the group. Thus we need only consider subsets of $\mathbb{Z}_4$ that contain [0] **and** that have size $1, 2$ or 4. Thus we look at the following subsets:

$$\{[0]\}, \{[0],[1]\}, \{[0],[2]\}, \{[0],[3]\}, \{[0],[1],[2],[3]\}.$$

The first and the last one are (as always) subgroups. Of the remaining three clearly only $\{[0],[2]\}$ passes the subgroup test. Thus all subgroups of $\mathbb{Z}_4$ are $\{[0]\}, \{[0],[2]\}, \{[0],[1],[2],[3]\}$.

Problem E:

(1) This is false: Let $A = \mathbb{R}$ and let $f(x) = 0 = g(x)$ for all $x \in \mathbb{R}$. Clearly neither f nor g is injective and yet $f \circ g(x) = f(g(x)) = f(0) = 0$ and similarly $g \circ f(x) = 0$ for all $x \in \mathbb{R}$ thus $f \circ g = g \circ f$.

(2) This is false. Cantor's Lemma states that for any set A holds $|A| < |P(A)|$. This holds then also for $A = P(X)$ for any hypothetic set X.

(3) This is false as the counter example $a = 2, b = 4$ shows since then $gcd(a, b) = 2$ while $gcd(a^2, b) = 4$.

(4) This is false as seen by the group $(\mathbb{Z}_4, +)$ and $g = [0]$. Then indeed $g^2 = [0] = e$ but $[1] + [1] \neq [0]$.