TWEEDE DEELTENTAMEN WISB 212

Analyse in Meer Variabelen

04–07–2006 14–17 uur

- *Zet uw naam en collegekaartnummer op elk blad alsmede het totaal aantal ingeleverde bladzijden.*
- *De verschillende onderdelen van het vraagstuk zijn zoveel als mogelijk is, onafhankelijk van elkaar. Indien u een bepaald onderdeel niet of slechts ten dele kunt maken, mag u de resultaten daaruit gebruiken bij het maken van de volgende onderdelen. Raak dus niet ontmoedigd indien het u niet lukt een bepaald onderdeel te maken en ga gewoon door.*
- *Bij dit tentamen mogen boeken, syllabi, aantekeningen en/of rekenmachine* **NIET** *worden gebruikt.*
- *De antwoorden mogen uiteraard in het Nederlands worden gegeven, ook al zijn de vraagstukken in het Engels geformuleerd.*
- *De drie vraagstukken tellen* **NIET** *evenzwaar: zij tellen voor 35, 25 en 40%, respectievelijk, van het totaalcijfer.*

Exercise 0.1 (Green's first identity by means of Gauss' Divergence Theorem). Consider $B^2 = \{ x \in \mathbf{R}^2 \mid \|x\| < 1 \}$ and $g : \mathbf{R}^2 \to \mathbf{R}$ given by $g(x) = x_1^2 - x_2^2$.

(i) Prove
$$\int_{B^2} \| \operatorname{grad} g(x)\|^2 \, dx = 2\pi.$$

(ii) Recall that $\frac{\partial g}{\partial \nu} = \langle \operatorname{grad} g, \nu \rangle$, the derivative in the direction of the outer normal ν to ∂B^2, and compute
$$\int_{\partial B^2} \Big(g \, \frac{\partial g}{\partial \nu} \Big)(y) \, d_1 y.$$

Hint: Use $2(\cos^2 \alpha - \sin^2 \alpha)^2 = 2\cos^2 2\alpha = 1 + \cos 4\alpha$.

The equality of the two integrals above is no accident, as we will presently show. To this end, suppose $h : \mathbf{R}^2 \to \mathbf{R}$ to be an arbitrary C^2 function. Note that $h \operatorname{grad} h : \mathbf{R}^2 \to \mathbf{R}^2$ is a C^1 vector field and recall the identity $\operatorname{div} \operatorname{grad} = \Delta$.

(iii) Prove $\operatorname{div}(h \operatorname{grad} h) = \| \operatorname{grad} h\|^2 + h \, \Delta h$.

(iv) Suppose $\Omega \subset \mathbf{R}^2$ satisfies the conditions of Gauss' Divergence Theorem. Apply this theorem to verify
$$(\star) \qquad \int_{\Omega} (h \, \Delta h)(x) \, dx + \int_{\Omega} \| \operatorname{grad} h(x)\|^2 \, dx = \int_{\partial \Omega} \Big(h \, \frac{\partial h}{\partial \nu} \Big)(y) \, d_1 y.$$

(v) Derive $(\star)$ in part (iv) directly from Green's first identity.

(vi) Show that the equality of the integrals in parts (i) and (ii) follows from $(\star)$ in part (iv).

Exercise 0.2 (Area of surface in $\mathbf{C}^2$). As usual, we identify $z = y_1 + i y_2 \in \mathbf{C}$ with $y = (y_1, y_2) \in \mathbf{R}^2$. In particular, an open set $D \subset \mathbf{C}$ is identified with the corresponding $D \subset \mathbf{R}^2$ and a complex-differentiable function $f : D \to \mathbf{C}$ with the vector field $f = (f_1, f_2) : D \to \mathbf{R}^2$. Thus, we will study $\operatorname{graph}(f) \subset \mathbf{C}^2$ in the form of the following set:

$$V = \{ (y, f(y)) \in \mathbf{R}^4 \mid y \in D \subset \mathbf{R}^2 \} = \operatorname{im}(\phi) \qquad \text{with}$$
$$\phi : D \to \mathbf{R}^4 \qquad \text{given by} \qquad \phi(y) = (y_1, y_2, f_1(y), f_2(y)).$$

It is obvious that V is a C^∞ submanifold in $\mathbf{R}^4$ of dimension 2 and that ϕ is a C^∞ embedding.

(i) Compute the Euclidean 2-dimensional density ω_ϕ on V determined by ϕ. Next, use the Cauchy–Riemann equations $D_1 f_1 = D_2 f_2$ and $D_1 f_2 = -D_2 f_1$ to show the following identity of functions on $\mathbf{R}^2$:
$$\omega_\phi = 1 + \| \operatorname{grad} f_1\|^2 = 1 + \| \operatorname{grad} f_2\|^2.$$

Suppose D to be a bounded open Jordan measurable set and deduce
$$\operatorname{vol}_2(V) = \operatorname{area}(D) + \int_D \| \operatorname{grad} f_1(y)\|^2 \, dy.$$

(ii) Suppose $D = \{ z \in \mathbf{C} \mid |z| < 1 \}$ and $f(z) = z^2$. Apply the preceding result as well as part (i) in Exercise 0.2 in order to establish that in this case we have $\operatorname{vol}_2(V) = 3\pi$.

Exercise 0.3 (Computation of $\zeta(2)$ by successive integration). Define the open set $J = \,]\,0, \sqrt{2}\,[\, \subset \mathbf{R}$ and the function $m : J \to \mathbf{R}$ by $m(y_1) = \min(y_1, \sqrt{2} - y_1)$.

(i) Sketch the graph of m. Verify that the open subset $\Diamond$ of $\mathbf{R}^2$ is a square of area 1 if we set

$$\Diamond = \{\, y \in \mathbf{R}^2 \mid y_1 \in J,\ -m(y_1) < y_2 < m(y_1)\,\}.$$

(ii) Define

$$f : \Diamond \to \mathbf{R} \qquad \text{by} \qquad f(y) = \frac{1}{2 - y_1^2 + y_2^2}.$$

Compute by successive integration

$$\int_{\Diamond} f(y)\, dy = \frac{\pi^2}{12}.$$

At $(\sqrt{2}, 0)$, which belongs to the closure in $\mathbf{R}^2$ of $\Diamond$, the integrand f is unbounded. Yet, without proof one may take the convergence of the integral for granted.
Hint: Write the integral the sum of two integrals, one involving $]\,0, \frac{1}{2}\sqrt{2}\,[$ and one $]\,\frac{1}{2}\sqrt{2}, \sqrt{2}\,[$, which can be computed to be $\frac{\pi^2}{36}$ and $\frac{\pi^2}{18}$, respectively. In doing so, use that $f(y) = f(y_1, -y_2)$. Furthermore, without proof one may use the following identities, which easily can be verified by differentiation:

$$\begin{aligned} \int f(y_1, y_2)\, dy_2 &= : g(y_1, y_2) := \frac{1}{\sqrt{2 - y_1^2}} \arctan\Big(\frac{y_2}{\sqrt{2 - y_1^2}}\Big), \\ \int g(y_1, y_1)\, dy_1 &= \frac{1}{2} \arctan^2\Big(\frac{y_1}{\sqrt{2 - y_1^2}}\Big), \\ \int g(y_1, \sqrt{2} - y_1)\, dy_1 &= -\arctan^2\Bigg(\sqrt{\frac{\sqrt{2} - y_1}{\sqrt{2} + y_1}}\Bigg). \end{aligned}$$

Introduce the open set $I = \,]\,0, 1\,[\, \subset \mathbf{R}$, and furthermore the counterclockwise rotation of $\mathbf{R}^2$ about the origin by the angle $\frac{\pi}{4}$ by

$$\Psi \in \mathrm{End}(\mathbf{R}^2) \qquad \text{with} \qquad \Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad \text{set} \qquad \square = I^2 \subset \mathbf{R}^2.$$

(iii) Show that $\Psi : \Diamond \to \square$ is a C^∞ diffeomorphism and using this fact deduce from part (ii)

$$\int_{\square} \frac{1}{1 - x_1 x_2}\, dx = \frac{\pi^2}{6}.$$

(iv) Conclude from part (iii)

$$\int_I \frac{\log(1 - x)}{x}\, dx = -\frac{\pi^2}{6}.$$

Give arguments that the integrand is a bounded continuous function on I near 0.

(v) Compute $\int_{\square} (x_1 x_2)^{k-1}\, dx$, for $k \in \mathbf{N}$. Assuming without proof that in this particular case summation of an infinite series and integration may be interchanged, use part (iii) (or part (iv)) to show Euler's celebrated identity

$$\zeta(2) := \sum_{k \in \mathbf{N}} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Solution of Exercise 0.1

(i) We have $\operatorname{grad} g(x) = 2(x_1, -x_2)$ and so $\|\operatorname{grad} g(x)\|^2 = 4\|x\|^2$. Introducing polar coordinates (r, α) in $\mathbf{R}^2 \setminus \{\, (x_1, 0) \in \mathbf{R}^2 \mid x_1 \le 0 \,\}$, which leads to a C^1 change of coordinates, we find

$$\int_{B^2} \|\operatorname{grad} g(x)\|^2 \, dx = \int_{-\pi}^{\pi} \int_0^1 4r^3 \, dr \, d\alpha = 2\pi [r^4]_0^1 = 2\pi.$$

(ii) $\partial B^2 = S^1$, which implies $\nu(y) = y$. Therefore

$$\Big(g \, \frac{\partial g}{\partial \nu}\Big)(y) = g(y) \langle\, 2(y_1, -y_2),\, (y_1, y_2) \,\rangle = 2g(y)^2.$$

Note $S^1 = \operatorname{im}(\phi)$ with $\phi(\alpha) = (\cos\alpha, \sin\alpha)$. Hence $\omega_\phi(\alpha) = \|(-\sin\alpha, \cos\alpha)\| = 1$ and so

$$\int_{\partial B^2} \Big(g \, \frac{\partial g}{\partial \nu}\Big)(y) \, d_1 y = \int_{-\pi}^{\pi} 2(\cos^2\alpha - \sin^2\alpha)^2 \, d\alpha = \int_{-\pi}^{\pi} (1 + \cos 4\alpha) \, d\alpha = 2\pi.$$

(iii) We have

$$\operatorname{div}(g \operatorname{grad} g) = \sum_{1 \le j \le 2} D_j (g \, D_j g) = \sum_{1 \le j \le 2} ((D_j g)^2 + g \, D_j^2 g) = \|\operatorname{grad} g\|^2 + g \, \Delta g.$$

(iv) The assertion follows from application of Gauss' Divergence Theorem 7.8.5 to the vector field $g \operatorname{grad} g$; indeed,

$$\begin{aligned} \int_\Omega \operatorname{div}(g \operatorname{grad} g)(x) \, dx &= \int_{\partial\Omega} \langle\, g(y) \operatorname{grad} g(y),\, \nu(y) \,\rangle \, d_1 y = \int_{\partial\Omega} g(y) \, \langle \operatorname{grad} g, \nu \,\rangle(y) \, d_1 y \\ &= \int_{\partial\Omega} \Big(g \, \frac{\partial g}{\partial \nu}\Big)(y) \, d_1 y. \end{aligned}$$

(v) Set $f = g$ in Green's first identity

$$\int_\Omega (g \, \Delta f)(x) \, dx = \int_{\partial\Omega} \Big(g \, \frac{\partial f}{\partial \nu}\Big)(y) \, d_{n-1} y - \int_\Omega \langle \operatorname{grad} f,\, \operatorname{grad} g \,\rangle(x) \, dx.$$

(vi) This follows from $\Delta g = 2 - 2 = 0$.

Solution of Exercise 0.2

(i) According to Lemma 8.3.10.(i) and (ii) the Cauchy–Riemann equations apply to the real and imaginary parts f_1 and f_2 of the holomorphic function f; consequently, we have the following equality of mappings $\mathbf{R}^2 \to \operatorname{Mat}(2, \mathbf{R})$:

$$\begin{aligned} (D\phi)^t \, D\phi &= \begin{pmatrix} 1 & 0 & D_1 f_1 & D_1 f_2 \\ 0 & 1 & D_2 f_1 & D_2 f_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ D_1 f_1 & D_2 f_1 \\ D_1 f_2 & D_2 f_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 + (D_1 f_1)^2 + (D_1 f_2)^2 & D_1 f_1 \, D_2 f_1 + D_1 f_2 \, D_2 f_2 \\ D_1 f_1 \, D_2 f_1 + D_1 f_2 \, D_2 f_2 & 1 + (D_2 f_1)^2 + (D_2 f_2)^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 + \|\operatorname{grad} f_1\|^2 & 0 \\ 0 & 1 + \|\operatorname{grad} f_1\|^2 \end{pmatrix}. \end{aligned}$$

Indeed, the coefficient of index $(2,1)$ equals $D_1 f_1\, D_2 f_1 - D_2 f_1\, D_1 f_1 = 0$. In view of Definition 7.3.1. – Theorem we obtain

$$\omega_\phi = \sqrt{\det\left((D\phi)^t\, D\phi\right)} = \sqrt{(1 + \|\,\mathrm{grad}\, f_1\|^2)^2} = 1 + \|\,\mathrm{grad}\, f_1\|^2.$$

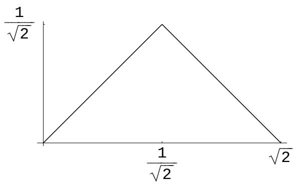
The last assertion now follows, because

$$\mathrm{vol}_2(V) = \int_V d_2 x = \int_D \omega_\phi(y)\, dy = \int_D (1 + \|\,\mathrm{grad}\, f_1(y)\|^2)\, dy.$$

(ii) $f_1(y) = \mathrm{Re}(y_1 + iy_2)^2 = y_1^2 - y_2^2 = g(y)$ with g as in Exercise 0.2. The assertion is a consequence from $\mathrm{area}(D) = \pi$ and part (i) of that exercise.

Solution of Exercise 0.3

(i) $\mathrm{graph}(m)$ is given by



This is an isosceles rectangular triangle of hypothenuse $\sqrt{2}$, hence its area equals $\frac{1}{2}$.

(ii) Note $J = \frac{1}{2}J \cup (\frac{1}{2}\sqrt{2} + \frac{1}{2}J)$ while the two subintervals have only one point in common. On $\frac{1}{2}J$ and $\frac{1}{2}\sqrt{2} + \frac{1}{2}J$ one has $m(y_1) = y_1$ and $m(y_1) = \sqrt{2} - y_1$, respectively. Furthermore $f(y) = f(y_1, -y_2)$. Therefore, using a generalization of Corollary 6.4.3 on interchanging the order of integration and the antiderivatives as given in the hint, one obtains

$$\begin{aligned}
\int_\Diamond f(y)\, dy &= 2\int_0^{\frac{1}{2}\sqrt{2}} \int_0^{y_1} f(y)\, dy_2\, dy_1 + 2\int_{\frac{1}{2}\sqrt{2}}^{\sqrt{2}} \int_0^{\sqrt{2}-y_1} f(y)\, dy_2\, dy_1 \\
&= 2\int_0^{\frac{1}{2}\sqrt{2}} g(y_1, y_1)\, dy_1 + 2\int_{\frac{1}{2}\sqrt{2}}^{\sqrt{2}} g(y_1, \sqrt{2} - y_1)\, dy_1 \\
&= \arctan^2\left(\frac{\sqrt{\frac{1}{2}}}{\sqrt{\frac{3}{2}}}\right) + 2\arctan^2\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi^2}{36} + \frac{\pi^2}{18} = \frac{\pi^2}{12},
\end{aligned}$$

because $\tan(\frac{\pi}{6}) = \frac{1}{\sqrt{3}}$.

(iii) The rotations Ψ and Ψ^{-1} are bijective and C^∞; hence, Ψ is a C^∞ diffeomorphism. From the description of Ψ as a specific rotation one gets $\Psi(\Diamond) = \Box$. Thus, $\Psi : \Diamond \to \Box$ is a C^∞ diffeomorphism. Observe that, for $y \in \Diamond$ and $x = \Psi(y) \in \Box$,

$$\frac{1}{1 - x_1 x_2} = \frac{1}{1 - \frac{1}{2}(y_1 - y_2)(y_1 + y_2)} = 2f(y) \qquad \text{and} \qquad |\det D\Psi(y)| = 1.$$

Application of the Change of Variables Theorem 6.6.1 now leads to the desired equality.

(iv) Note that

$$\int_I \frac{1}{1-x_1x_2}\,dx_2 = \Big[-\frac{\log(1-x_1x_2)}{x_1}\Big]_0^1 = -\frac{\log(1-x_1)}{x_1}.$$

Since $\square = I \times I$, one obtains the desired formula by means of Corollary 6.4.3 once more. Taylor series expansion of the integrand about 0 shows that it equals $-1+\mathcal{O}(x)$, for $x \downarrow 0$.

(v) Obviously

$$\int_\square x_1^{k-1}x_2^{k-1}\,dx = \Big(\int_I x^{k-1}\,dx\Big)^2 = \frac{1}{k^2}.$$

Summation of the geometric series leads to

$$\sum_{k\in\mathbf{N}}(x_1x_2)^{k-1} = \frac{1}{1-x_1x_2}.$$

Integrating the equality over $\square$ and interchanging summation of an infinite series and integration one finds, on the basis of part (iii)

$$\sum_{k\in\mathbf{N}}\frac{1}{k^2} = \sum_{k\in\mathbf{N}}\int_\square (x_1x_2)^{k-1}\,dx = \int_\square \frac{1}{1-x_1x_2}\,dx = \frac{\pi^2}{6}.$$

Background. Compare this exercise with Exercise 6.39. Note that the definition of the integral in part (ii) needs some care, as the integrand f becomes infinite at the corner $(\sqrt{2},0)$ of the closure of $\Diamond$. Since f is continuous and positive on the open set $\Diamond$, in order to prove convergence of the integral it suffices to show the existence of an increasing sequence of compact Jordan measurable sets $K_k \subset \Diamond$ such that $\cup_{k\in\mathbf{N}}K_k = \Diamond$ and that the $\int_{K_k} f(y)\,dy$ exist and converge as $k \to \infty$, see Theorem 6.10.6. One may do this, by choosing the subsets K_k to be the closures of the contracted squares $\frac{k-1}{k}\,\Diamond$.

Next, the antiderivatives in part (ii) may be computed as follows. For the first one, write

$$f(y) = \frac{1}{\sqrt{2-y_1^2}}\,\frac{1}{1+\big(\frac{y_2}{\sqrt{2-y_1^2}}\big)^2}\,\frac{d}{dy_2}\frac{y_2}{\sqrt{2-y_1^2}} \qquad \text{and set} \qquad u = u(y_2) = \frac{y_2}{\sqrt{2-y_1^2}};$$

further, use $\int \frac{1}{1+u^2}\,du = \arctan u$. For the second antiderivative, apply the change of variables

$$v = v(y_1) = \frac{y_1}{\sqrt{2-y_1^2}}, \quad \text{so} \quad y_1 = \sqrt{2}\,\frac{v}{\sqrt{1+v^2}}, \qquad \sqrt{2-y_1^2} = \frac{\sqrt{2}}{(1+v^2)^{\frac{1}{2}}}, \qquad \frac{dy_1}{dv} = \frac{\sqrt{2}}{(1+v^2)^{\frac{3}{2}}}.$$

Thus,

$$\int g(y_1,y_1)\,dy_1 = \int \frac{\arctan v}{1+v^2}\,dv = \frac{1}{2}\arctan^2 v.$$

For the third antiderivative, apply the change of variables

$$w = w(y_1) = \frac{\sqrt{2}-y_1}{\sqrt{2-y_1^2}}, \quad \text{so} \quad y_1 = \sqrt{2}\,\frac{1-w^2}{1+w^2}, \qquad \sqrt{2-y_1^2} = \frac{2\sqrt{2}w}{1+w^2}, \qquad \frac{dy_1}{dv} = -\frac{4\sqrt{2}w}{(1+w^2)^2}.$$

Thus,

$$\int g(y_1,y_1)\,dy_1 = -2\int \frac{\arctan w}{1+w^2}\,dv = -\arctan^2 w.$$