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blok 3, 2014/ 2015
May 29, 2015

Analyse in meer variabelen, WISB212

Hertentamen

Family name: ____________________ Given name: ____________________

Student number: ____

Please:

- Switch off your mobile phone and put it into your bag.
- Write with a blue or black pen, not with a green or red one, nor with a pencil.
- Write your name on each sheet.
- Hand in this sheet, as well.
- Hand in only one solution to each problem.

The examination time is 180 minutes.

You are not allowed to use books, calculators, or lecture notes, but you may use 1 sheet of handwritten personal notes (A4, both sides).

You may use the following without proof:

- the theorems, propositions, and corollaries that were proved in the lecture or in the book, unless otherwise stated
- The composition of two smooth maps is smooth.
- smoothness of a map that is given by an "explicit formula" (if the map is indeed smooth)
- The graph of a continuous function defined on a compact set is negligible.

Prove every other statement you make. Justify your calculations. Check the hypotheses of the theorems you are using.

If you are not able to solve one part of a problem, try to solve the other parts.

You may write in Dutch.

31 points will yield a passing grade 6, and 65 points a grade 10.

Good luck!

1	2	3	4	5	6	7	8	$\sum$
/5	/7	/8	/9	/10	/7	/5	/14	/65

Problem 1 (differentiability of components, 5 pt)**.** Let $U \subseteq \mathbb{R}^n$ be an open subset, $f : U \to \mathbb{R}^p$ a map, and $x_0 \in U$.

(i) Show that f is differentiable at x_0 if and only if the i-th component f^i is differentiable at x_0 for every $i = 1, \ldots, p$.

(ii) Find a formula for $Df(x_0)$ in terms of $D(f^i)(x_0)$ in this case.

Remark: This was a proposition in the book by Duistermaat and Kolk. You need to prove this result here.

Problem 2 (nonlinear equation, 7 pt)**.** (i) Prove that there exist numbers $a > 0$ and $b > 0$ with the following properties: For every $y \in \big(1 - b, 1 + b\big)$ there exists a unique solution $x = x_y \in (-a, a)$ of the equation

$$\sin\big(x + \pi y\big) = x.$$

Furthermore, the function $y \mapsto x_y$ is smooth.

(ii) Calculate the derivative of this function at 1.

Problem 3 (curve in the plane, 8 pt)**.** (i) Draw a picture of the set

$$M := \big\{x \in \mathbb{R}^2 \,\big|\, 5x_1^2 + 5x_2^2 - 6x_1x_2 = 4\big\}, \tag{1}$$

indicating four points that lie on it.

(ii) Prove that this set is a submanifold of $\mathbb{R}^2$. Calculate its dimension.

(iii) Compute the tangent space of M at any point $x \in M$.

Problem 4 (minimum, 9 pt)**.** Let M be as in (1).

(i) Prove that the function

$$f : M \to \mathbb{R}, \quad f(x) := x_1 - x_2,$$

attains its minimum on M.

(ii) Calculate the minimum of f on M.

Remarks: You may use results from WISB111 (Inleiding Analyse), and the fact that every minimum point of a function f is a critical point for f.

Problem 5 (volume of distorted simplex, 10 pt)**.** (i) For $n = 1, 2, 3$ draw the set

$$\Delta_n := \left\{x \in \mathbb{R}^n \,\middle|\, x_1, \ldots, x_n \geq 0,\ \sum_{i=1}^{n} \frac{x_i}{i} \leq 1\right\},$$

indicating its corner points.

(ii) Prove that Δ_n is Jordan-measurable for every $n \in \mathbb{N}$.

(This problem continues on the reverse page.)

(iii) Calculate the Jordan-measure of Δ_n.

Remark: You may use the fact that for every Jordan-measurable set $S \subseteq \mathbb{R}^m$ and every $c \geq 0$ the set

$$cS = \{cx \,|\, x \in \mathbb{R}^m\}$$

is Jordan-measurable with Jordan-measure

$$|cS| = c^m |S|.$$

Problem 6 (area of spherical cap, 7 pt)**.** Let $a \in (0, 1)$.

(i) Draw the spherical cap

$$\{x \in S^2 \,|\, x_3 \geq a\}.$$

(ii) Calculate its area, i.e., 2-dimensional volume.

Problem 7 (flux through hemisphere, 5 pt)**.** Consider the upper hemisphere

$$M := \{x \in S^2 \,|\, x_3 \geq 0\},$$

the vector field

$$X : \mathbb{R}^3 \to \mathbb{R}^3, \quad X(x) := \arctan(x_3) \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix},$$

and the unit normal vector field

$$\nu : M \to \mathbb{R}^3, \quad \nu(x) := x.$$

Calculate the flux (= surface-integral) of the vector field

$$\nabla \times X = \mathbf{curl}\, X : \mathbb{R}^3 \to \mathbb{R}^3$$

through M with respect to ν.

Problem 8 (limit of Riemann integrable functions, 14 pt)**.** Let $n \in \mathbb{N}$, $f : R := [0, 1]^n \to \mathbb{R}$ be a function, and for $i \in \mathbb{N}$, let $f_i : R \to \mathbb{R}$ be a (properly) Riemann integrable function. **Prove or disprove** each of the following statements.

(i) The function f is Riemann integrable if $f_i(x)$ converges to $f(x)$, as $i \to \infty$, for every $x \in R$.

(ii) The function f is Riemann integrable if

$$\sup_{x \in R} \big|f_i(x) - f(x)\big| \to 0, \quad \text{as } i \to \infty.$$

Remark: In this problem you may use any exercise from the assignments (and any theorem from the lecture and the book by Duistermaat and Kolk).