F. Ziltener

blok 3, 2014/ 2015
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Analyse in meer variabelen, WISB212

Tentamen

Family name: ______________________ Given name: ______________________

Student number: ____

Please:

- Switch off your mobile phone and put it into your bag.
- Write with a blue or black pen, not with a green or red one, nor with a pencil.
- Write your name on each sheet.
- Hand in this sheet, as well.
- Hand in only one solution to each problem.

The examination time is 180 minutes.

You are not allowed to use books, calculators, or lecture notes, but you may use 1 sheet of handwritten personal notes (A4, both sides).

You may use the following without proof:

- the theorems that were proved in the lecture or in the book, unless otherwise stated
- The composition of two smooth maps is smooth.
- smoothness of a map that is given by an "explicit formula" (if the map is indeed smooth)
- a formula for the derivative of a multilinear map
- If two maps $f : U \to V$ and $g : U \to W$ are differentiable at $x \in U$ then $(f, g) : U \to V \times W$ is differentiable at x.
- a formula for $D(f, g)(x)$

Prove every other statement you make. Justify your calculations. Check the hypotheses of the theorems you are using.

If you are not able to solve one part of a problem, try to solve the other parts.

You may write in Dutch.

31 points will yield a passing grade 6, and 66 points a grade 10.

Good luck!

1	2	3	4	5	6	7	8	9	$\sum$
/5	/6	/9	/7	/5	/4	/12	/4	/19	/71

We denote by $\mathbb{R}^{n_1 \times n_2}$ the space of real $n_1 \times n_2$-matrices.

Problem 1 (Leibniz Rule, 5 pt)**.** Let $U \subseteq \mathbb{R}^m$ be an open subset, $x_0 \in U$, and $f : U \to \mathbb{R}^{n_1 \times n_2}$ and $g : U \to \mathbb{R}^{n_2 \times n_3}$ be maps that are differentiable at x_0. Prove that the map

$$fg : U \to \mathbb{R}^{n_1 \times n_3}$$

is differentiable at x_0 and compute its derivative at this point. Here $(fg)(x) := f(x)g(x)$ is the product of the matrices $f(x)$ and $g(x)$, for every $x \in U$.

Problem 2 (root of a matrix, 6 pt)**.** Let $n, k \in \mathbb{N}$. We denote by $\mathbf{1} \in \mathbb{R}^{n \times n}$ the identity matrix. Prove that there exist open neighbourhoods U and $V \subseteq \mathbb{R}^{n \times n}$ of $\mathbf{1}$ with the following properties. For every $A \in U$ there exists a unique solution $B = B_A \in V$ of the equation

$$B^k = A.$$

Furthermore, the map $A \mapsto B_A$ is smooth.

Problem 3 (curve given as level set, 9 pt)**.** (i) Draw the set

$$M := \big\{ x \in \mathbb{R}^2 \,\big|\, x_1^4 + 16 x_2^4 = 1 \big\},$$

indicating 8 points that lie on M.

(ii) Prove that M is a smooth submanifold of $\mathbb{R}^2$. Calculate its dimension.

(iii) Calculate the tangent space to M at every point $x \in M$.

Problem 4 (curve given by parametrization, 7 pt)**.** We define

$$\psi : \mathbb{R} \to \mathbb{R}^2, \quad \psi(y) := \big(y + y^5, -y - y^7 \big).$$

(i) Draw a picture of the image M of ψ, indicating 3 points that lie on M.

(ii) Show that M is a smooth submanifold of $\mathbb{R}^2$. Calculate its dimension.

Problem 5 (Lagrange multiplier method, 5 pt)**.** Let

$$n, p \in \mathbb{N}, \quad U \subseteq \mathbb{R}^n \text{ open}, \ F \in C^1(U, \mathbb{R}), \quad g \in C^1(U, \mathbb{R}^p),$$
$$M := g^{-1}(0), \quad f := F|_M, \quad x_0 \in M,$$
$$L : U \times \mathrm{Lin}\big(\mathbb{R}^p, \mathbb{R}\big) \to \mathbb{R}, \quad L(x, \lambda) := F(x) - \lambda g(x).$$

Assume that g is a submersion and there exists $\lambda \in \mathrm{Lin}(\mathbb{R}^p, \mathbb{R})$ such that (x_0, λ) is a critical point of L. Show that x_0 is a critical point of f.

Problem 6 (two-dimensional integral, 4 pt)**.** Calculate

$$\int_{-1}^{1} \left(\int_0^1 \left(e^{y \sin x} - e^{-y \sin x} \right) dx \right) dy.$$

Problem 7 (integral of rotationally invariant function, 12 pt)**.** Let $r_0 > 0$ and $\widetilde{f} : [0, r_0] \to \mathbb{R}$ be a continuous function. We define

$$f : \overline{B}^2_{r_0} \to \mathbb{R}, \quad f(x) := \widetilde{f}(\|x\|).$$

Find a formula for $\int_{\overline{B}^2_{r_0}} f(x)\, dx$ in terms of $\widetilde{f}$. Justify your computation.

Remark: This problem was a corollary in the lecture. You are asked to prove this corollary here.

Problem 8 (integral over ball, 4 pt)**.** Calculate the integral

$$\int_{B^2} \big(D_1 X^2 - D_2 X^1\big)(x)\, dx$$

for

$$X(x) := \arctan\big(\|x\|^2\big) \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}.$$

Problem 9 (Riemann-integrable function, 19 pt)**.** Let $n \in \mathbb{N}$. Does there exist a Riemann integrable function $f : [0,1]^n \to \mathbb{R}$ such that for every open subset $\emptyset \neq U \subseteq (0,1)^n$ the set

$$\big\{ x \in U \,\big|\, f \text{ is discontinuous at } x \big\}$$

is uncountable?

Hint: First try to find an f for which the above set is nonempty for every U.

Remark: In this problem you may use any exercise from the assignments (and any theorem from the lecture and the book by Duistermaat and Kolk).