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blok 3, 2013/ 2014
April 15, 2014

Analyse in meer variabelen, WISB212

Tentamen

Family name: ________________ Given name: ________________

Student number: ________________________________

Please:

- Switch off your mobile phone and put it into your bag.
- Write with a blue or black pen, not with a green or red one, nor with a pencil.
- Write your name on each sheet.
- Hand in this sheet, as well.
- Hand in only one solution to each problem.

The examination time is 180 minutes.

You are not allowed to use books, calculators, or lecture notes, but you may use 1 sheet of handwritten personal notes (A4, both sides).

Unless otherwise stated, you may use results from the lecture and the book without proving them.

You may use the fact that a map given by one "explicit formula" is smooth.

Prove every other statement you make. Justify your calculations. Check the hypotheses of the theorems you use.

Even if you are not able to solve one part of a problem, try to solve the other parts.

You may write in Dutch.

30 points will yield a passing grade 6, and 63 points a grade 10.

Good luck!

1	2	3	4	5	6	7	8	9	10	$\sum$
/7	/5	/4	/4	/10	/8	/4	/4	/7	/10	/63

Problem 1 (nonlinear system of equations, 7pt). (i) Prove that there exist numbers $r, r' > 0$ with the following properties: For every $y \in \mathbb{R}^2$ satisfying $\|y - (1,1)\| < r'$ there exists a unique solution $x = x_y \in \mathbb{R}^2$ of the equations

$$\begin{aligned} \sin(x_1) + e^{-x_2} &= y_1 \\ e^{x_1} + \sin(x_2) &= y_2, \end{aligned}$$

satisfying $\|x_y\| < r$. Furthermore, the map $y \mapsto x_y$ is smooth.

(ii) Calculate the derivative of the map $y \mapsto x_y$ at the point $(1,1)$.

Problem 2 (tangent space to linear subspace, 5pt). Let M be a d-dimensional linear subspace of $\mathbb{R}^n$ and $x_0 \in M$. Determine the tangent space of M at x_0.

Remark: You may use the fact that M is a smooth submanifold of $\mathbb{R}^n$.

Problem 3 (a maximum is a critical point, 4pt). Let $M \subseteq \mathbb{R}^n$ be a C^1-submanifold, $f \in C^1(M, \mathbb{R})$, and $x_0 \in M$ be a point at which f attains its maximum. Show that x_0 is a critical point for f.

Hint: You may use a similar statement for a function of one real variable.

Problem 4 (two-dimensional integral, 4pt). Calculate

$$\int_{-1}^{1} \left(\int_0^1 \sin\left(\cos(x_1) x_2\right) dx_1 \right) dx_2.$$

Problem 5 (volume of simplex, 10pt). (i) Draw the set

$$\Delta_n := \left\{ x \in \mathbb{R}^n \,\middle|\, x_1, \ldots, x_n \geq 0,\ x_1 + \cdots + x_n \leq 1 \right\}$$

for $n = 1, 2, 3$.

(ii) Let $n \in \mathbb{N}$. Prove that Δ_n is Jordan-measurable.

Hint: You may use one of the other problems of this exam.

(iii) Calculate the Jordan-measure of Δ_n.

Remark: You may use the fact that for every Jordan-measurable set $S \subseteq \mathbb{R}^m$ and every $c \geq 0$ the set

$$cS = \left\{ cx \,\middle|\, x \in \mathbb{R}^m \right\}$$

is Jordan-measurable with Jordan-measure

$$|cS| = c^m |S|.$$

Problem 6 (bitten piece of cake, 8pt). (i) Draw a picture of the set

$$S := \{x \in [0,\infty) \times [0,\infty) \mid x_1 \geq x_2,\, 1 \leq \|x\| \leq 2\}.$$

(ii) Show that the function

$$f : S \to \mathbb{R}, \quad f(x) := \|x\|,$$

is Riemann-integrable over S, and calculate the integral

$$\int_S f(x)\, dx.$$

Remarks: You may use the fact that S is Jordan-measurable.

Check all hypotheses of each theorem you use.

Problem 7 (two-dimensional integral, 4pt). Calculate the integral

$$\int_{B_1^2} \big(D_1 X^2 - D_2 X^1\big)(x)\, dx$$

for

$$X(x) := e^{\|x\|^2 - 1} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

Here B_1^2 denotes the open unit ball in $\mathbb{R}^2$.

Problem 8 (line integral, 4pt). Consider the ellipse

$$C := \{x \in \mathbb{R}^2 \mid x_1^2 + 9x_2^2 = 1\}.$$

(i) Calculate a unit tangent vector field T along C.

(ii) Calculate the line integral

$$\int_{C,T} X \cdot d\mathbf{s} = \int_C X \cdot T\, ds$$

for the vector field

$$X(x) := \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

Problem 9 (graph negligible, 7pt). Let $n \in \mathbb{N}$, $K \subseteq [0,1]^n$ be compact, and $f : K \to \mathbb{R}$ be continuous. Show that the graph of f is a negligible subset of $\mathbb{R}^{n+1}$.

Remark: This problem was a corollary in the book. You need to prove this result here.

Problem 10 (orthogonal matrices, 10pt). We denote by

$$\mathrm{O}(n) := \{A \in \mathbb{R}^{n \times n} \mid A^T A = \mathbf{1}\}$$

the set of orthogonal matrices.

(i) Prove that $\mathrm{O}(n)$ is a smooth submanifold of $\mathbb{R}^{n \times n} = \mathbb{R}^{(n^2)}$.

(ii) Calculate its dimension.

(iii) Determine the tangent space to $\mathrm{O}(n)$ at the identity matrix $\mathbf{1}$.

Remarks: This problem was a proposition and an example in the lecture. The proof of the proposition used a remark about matrices. You need to prove these results here.
You may use the formula for the derivative of a multilinear map.