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blok 3, 2013/ 2014
March 18, 2014

Analyse in meer variabelen, WISB212

Deeltentamen

Family name: ______________________ Given name: ______________________

Student number: ____

Please:

- Switch off your mobile phone and put it into your bag.
- Write with a blue or black pen, not with a green or red one, nor with a pencil.
- Write your name on each sheet.
- Hand in only one solution to each problem.

The examination time is 90 minutes.

You are not allowed to use books, calculators, lecture notes, or personal notes.

You may use theorems from the lecture and the book without proving them.

Prove every other statement you make.

Good luck!

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Problem 1 (derivative of bilinear map, 7pt)**.** Let $n_1, n_2 \in \mathbb{N}$ and

$$f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$$

be a bilinear map. Show that f is (totally) differentiable and calculate its derivative.

Problem 2 (hyperboloid, 9pt)**.** (i) Draw a picture of the hyperboloid

$$M := \{x \in \mathbb{R}^3 \,|\, x_1^2 + x_2^2 = x_3^2 + 1\}.$$

(ii) Prove that M is a smooth submanifold of $\mathbb{R}^3$ and calculate its dimension.

(iii) Calculate the tangent space to M at any point $x \in M$.

Problem 3 (maximum, 11pt)**.** Consider the curve

$$M := \{x \in \mathbb{R}^2 \,|\, x_1^4 + x_2^4 = 1\}$$

and the function

$$f : M \to \mathbb{R}, \quad f(x) := x_1 + x_2.$$

(i) Draw a picture of M and several level sets of f.

(ii) Prove that f attains its maximum on M.

(iii) Using the Method of Lagrange Multipliers, calculate the maximum of f on M. Show that the hypotheses of the theorem are indeed satisfied and make sure that your argument is logical.

Remarks: You may use results from WISB111 (Inleiding Analyse) and the fact that a maximum of a function $f \in C^1(M, \mathbb{R})$ is a critical point for f, without proof.

Problem 4 (smooth dependence of simple eigenvalue, 5pt)**.** Let $A_0 \in \mathbb{R}^{n \times n}$ and $\lambda_0 \in \mathbb{R}$ be an algebraically simple eigenvalue of A_0. This means that it is a simple zero of the characteristic polynomial $p(\lambda) := \det\big(\lambda \mathbf{1} - A_0\big)$. Prove that there exist open neighbourhoods $U \subseteq \mathbb{R}$ of λ_0 and $V \subseteq \mathbb{R}^{n \times n}$ of A_0, such that every $A \in V$ has a unique eigenvalue λ_A in U, and the map

$$V \ni A \mapsto \lambda_A \in U$$

is smooth.

Remark: You may use the facts that the determinant map is smooth and that the composition of smooth maps is smooth.