MIDTERM MULTIDIMENSIONAL REAL ANALYSIS

APRIL 16 2013, 13:30-16:30

- Put your name and studentnummer on every sheet you hand in.
- When you use a theorem, show that the conditions are met.
- You can give your answers either in English or in Dutch.
- The exam consists of three exercises and amounts for 40% of the total grade.

Exercise 1. ***(30 pt)*** In this exercise, we will compute the total derivative of the *inversion mapping* $G : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ defined by

$$G(x) = \frac{1}{\|x\|^2} x, \tag{1}$$

where $\|x\|$ is the standard norm in $\mathbb{R}^n$, i.e. $\|x\|^2 = \langle x, x\rangle = x^{\mathrm{T}} x$.

(a) (*5 pt*) Describe the action of the mapping (1) geometrically.

(b) (*10 pt*) Let $U \subset \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}$ and $G : U \to \mathbb{R}^n$ be two differentiable mappings. Define $fG : U \to \mathbb{R}^n$ via $(fG)(x) = f(x)G(x),\ x \in U$. Prove that fG is differentiable and

$$D(fG)(x) = f(x)DG(x) + G(x)Df(x), \quad x \in U. \tag{2}$$

(c) (*5 pt*) Using (2) with $f(x) = \|x\|^2$, compute the total derivative $DG(x)$ of the mapping (1) for $x \in U$, where $U = \mathbb{R}^n \setminus \{0\}$.

(d) (*10 pt*) Show that for $x \in U$ holds $DG(x) = \|x\|^{-2} A(x)$, where $A(x)$ is represented by an orthogonal matrix, i.e. $A^{\mathrm{T}}(x)A(x) = I$.

Exercise 2 ***(30 pt)*****.** Let $a, b, c > 0$ and let M be the *ellipsoid* in $\mathbb{R}^3$ defined as

$$M = \left\{ x \in \mathbb{R}^3 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1 \right\}.$$

(a) (*10 pt*) Find the tangent space of M at $x \in M$.

(b) (*20 pt*) Compute the distance from the origin to the geometric tangent plane to M at an arbitrary point $x \in M$.

Turn the page!

Exercise 3. ***(40 pt)*** Here, we will study a representation of the *Möbius Strip* in $\mathbb{R}^3$.

(a) (*5 pt*) Let $D = \{(\theta, t) \in \mathbb{R}^2 : -\pi < \theta < \pi, -1 < t < 1\}$ and let $\Phi : D \to \mathbb{R}^3$ be defined by

$$\Phi(\theta, t) = \begin{pmatrix} \left(2 + t\cos\left(\frac{\theta}{2}\right)\right)\cos\theta \\ \left(2 + t\cos\left(\frac{\theta}{2}\right)\right)\sin\theta \\ t\sin\left(\frac{\theta}{2}\right) \end{pmatrix}.$$

Prove that Φ is an immersion at any point in D.

(b) (*10 pt*) Show that $\Phi : D \to \Phi(D)$ is invertible and that the inverse mapping is continuous. Use this to conclude that $V = \Phi(D)$ is a C^∞ submanifold in $\mathbb{R}^3$ of dimension 2.

(c) (*5 pt*) Prove that any point $x \in V$ satisfies $g(x) = 0$, where $g : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$g(x) = 4x_2 + 4x_1x_3 - x_2(x_1^2 + x_2^2 + x_3^2) + 2x_3(x_1^2 + x_2^2). \qquad (3)$$

(d) (*10 pt*) The Möbius strip is the closure $M = \overline{V}$ of V in $\mathbb{R}^3$. Verify that the circle $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 4 \text{ and } x_3 = 0\}$ belongs to M. Give a parametrization of S by $\theta \in]-\pi, \pi]$. Prove that g introduced by (3) is a submersion at any point $x \in S$ except for $x = (-2, 0, 0)$.

(e) (*10 pt*) Show that $n_0 = (0, 0, 1) \in \mathbb{R}^3$ is orthogonal to the tangent space $T_{\Phi(0,0)}V$. Compute a continuous vector-valued function $n :]-\pi, \pi[\to \mathbb{R}^3$ such that $n(0) = n_0$ and for all $-\pi < \theta < \pi$ the vector $n(\theta) \in \mathbb{R}^3$ is orthogonal to $T_{\Phi(\theta,0)}V$ while $\|n(\theta)\| = 1$. Verify that

$$\lim_{\theta \to \pi} n(\theta) = -\lim_{\theta \to -\pi} n(\theta).$$

(f) (Bonus: *5 pt*) Sketch the set M and describe its geometry.