Tweede DEELTENTAMEN WISB 212
Analyse in Meer Variabelen

04–07–2005 9–12 uur

- *Zet uw naam en collegekaartnummer op elk blad, en op het eerste blad het totaal aantal ingeleverde bladzijden.*
- *De verschillende onderdelen van het vraagstuk zijn zoveel als mogelijk is, onafhankelijk van elkaar. Indien u een bepaald onderdeel niet of slechts ten dele kunt maken, mag u de resultaten daaruit gebruiken bij het maken van de volgende onderdelen. Raak dus niet ontmoedigd indien het u niet lukt een bepaald onderdeel te maken en ga gewoon door.*
- *De vraagstukken tellen* **NIET** *evenzwaar: Vraagstuk 1 telt voor 80 punten en Vraagstuk 2 voor 20 punten.*
- *De antwoorden mag u uiteraard in het Nederlands geven, ook al zijn de vraagstukken in het Engels geformuleerd.*
- *Bij dit tentamen mogen syllabi, aantekeningen en/of rekenmachine* **NIET** *worden gebruikt.*

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Exercise 1.1 (Two-step recurrences for hyperarea and volume). Write S^{n-1} and B^n for the unit sphere and the interior of the unit ball in $\mathbf{R}^n$, respectively, and set

$$a_{n-1} = \mathrm{hyperarea}_{n-1}(S^{n-1}) \qquad \text{and} \qquad v_n = \mathrm{vol}_n(B^n).$$

Here is a table of these numbers for low values of n:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
a_{n-1}	2	2π	4π	$2\pi^2$	$\frac{8\pi^2}{3}$	π^3	$\frac{16\pi^3}{15}$	$\frac{\pi^4}{3}$	$\frac{32\pi^4}{105}$	$\frac{\pi^5}{12}$	$\frac{64\pi^5}{945}$	$\frac{\pi^6}{60}$	$\frac{128\pi^6}{10395}$	$\frac{\pi^7}{360}$
v_n	2	π	$\frac{4\pi}{3}$	$\frac{\pi^2}{2}$	$\frac{8\pi^2}{15}$	$\frac{\pi^3}{6}$	$\frac{16\pi^3}{105}$	$\frac{\pi^4}{24}$	$\frac{32\pi^4}{945}$	$\frac{\pi^5}{120}$	$\frac{64\pi^5}{10395}$	$\frac{\pi^6}{720}$	$\frac{128\pi^6}{135135}$	$\frac{\pi^7}{5040}$

(i) In the table we see $a_{n-1} = n\, v_n$, for $1 \leq n \leq 14$. Prove this identity for all $n \in \mathbf{N}$, for instance, by applying Gauss' Divergence Theorem.

The table also suggests that the powers of π are given by the integral part of half the dimension and, furthermore, that there exist two-step recurrences

$$(\star) \qquad a_{n-1} = \frac{2\pi}{n-2}\, a_{n-3} \qquad \text{and} \qquad v_n = \frac{2\pi}{n}\, v_{n-2}.$$

In the following we will prove these identities geometrically (that is, without analyzing values of the Gamma function), for all $n \in \mathbf{N}$ sufficiently large. To this end, define the function $s : B^{n-2} \to \mathbf{R}_+$ by $s(x) = \sqrt{1 - \|x\|^2}$ and the mapping

$$\phi : D := B^{n-2} \times \,]-\pi, \pi\,[\;\to \mathbf{R}^n \qquad \text{by} \qquad \phi(x, \alpha) = \begin{pmatrix} x \\ s(x)\cos\alpha \\ s(x)\sin\alpha \end{pmatrix}.$$

(ii) Firstly, consider the case of $n = 3$. Prove that ϕ is injective and that $\mathrm{im}(\phi) = S^2$ except for a set which is negligible for 2-dimensional integration. Note that ϕ induces the mapping

$$\psi : C^2 := B^1 \times S^1 \to S^2 \qquad \text{given by} \qquad \psi(x, y) = \phi(x, \arg(y)) = \begin{pmatrix} x \\ s(x) y_1 \\ s(x) y_2 \end{pmatrix}.$$

Show that ψ is a bijection between the cylinder C^2 and the sphere minus two points. Furthermore, describe ψ in geometric terms, that is, as a projection (the inverse of ψ is known as *Lambert's cylindrical projection* of the sphere onto a tangent cylinder, see the next page for an illustration).

(iii) Next, consider the case of general $n \geq 3$. Prove $D_j s(x) = -\frac{x_j}{s(x)}$, for $1 \leq j \leq n-2$ and $x \in B^{n-2}$. Furthermore, write I_{n-2} for the identity matrix in $\mathrm{Mat}(n-2, \mathbf{R})$ and also x^t for the row vector obtained from $x \in B^{n-2}$ by means of transposition. Show that, for all $(x, \alpha) \in D$,

$$D\phi(x, \alpha) \in \mathrm{Lin}(\mathbf{R}^{n-1}, \mathbf{R}^n) \qquad \text{and} \qquad D\phi(x, \alpha)^t\, D\phi(x, \alpha) \in \mathrm{End}(\mathbf{R}^{n-1})$$

has the following matrix, respectively:

$$\begin{pmatrix} I_{n-2} & 0_{n-2} \\ -\dfrac{\cos\alpha}{s(x)} x^t & -s(x)\sin\alpha \\ -\dfrac{\sin\alpha}{s(x)} x^t & s(x)\cos\alpha \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} I_{n-2} + \dfrac{1}{s(x)^2} x x^t & 0 \\ 0^t & s(x)^2 \end{pmatrix}.$$

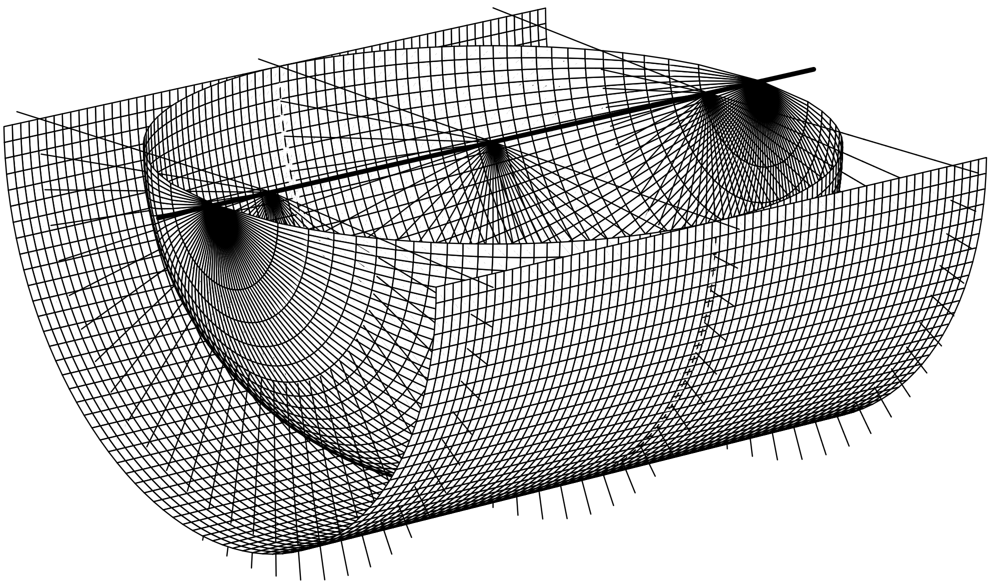

Illustration for part (ii): Lambert's projection from sphere onto tangent cylinder

(iv) Generalize the results from part (ii). Specifically, applying results from part (iii), verify that ϕ is a C^∞ embedding having an open part of S^{n-1} with negligible complement as an image.

(v) By considering the behavior of the following determinant (see part (iii)) under rotations of the element $x \in B^{n-2}$, show

$$\det\Big(I_{n-2} + \frac{1}{s(x)^2} x x^t\Big) = \frac{1}{s(x)^2} \qquad \text{and deduce} \qquad \omega_\phi(x, \alpha) = 1,$$

where ω_ϕ is the Euclidean density function associated with $\phi : D \to S^{n-1}$.

(vi) On the basis of parts (v) and (i) prove the first equality in $(\star)$ and then deduce the second one. In particular, prove by mathematical induction over $n \in \mathbf{N}$

$$v_{2n} = \frac{\pi^n}{n!}, \qquad v_{2n-1} = \frac{2^{2n}\,\pi^{n-1}\,n!}{(2n)!} \qquad \text{and} \qquad a_{2n-1} = \frac{2\pi^n}{(n-1)!}.$$

Next, we use the formula for v_{2n} in order to compute the volume of the standard $(n+1)$-tope Δ^n in $\mathbf{R}^n$ given by

$$\Delta^n = \{\, y \in \mathbf{R}^n_+ \mid \sum_{1 \le j \le n} y_j < 1 \,\}. \qquad \text{In fact, we claim} \qquad (\star\star) \qquad \mathrm{vol}_n(\Delta^n) = \frac{1}{n!}.$$

For proving this, introduce

$$\Psi : \Delta^n \times \,]-\pi, \pi[\,^n \to B^{2n} \qquad \text{with} \qquad \Psi(y, \alpha) = \begin{pmatrix} \sqrt{y_1}\cos\alpha_1 \\ \sqrt{y_1}\sin\alpha_1 \\ \vdots \\ \sqrt{y_n}\cos\alpha_n \\ \sqrt{y_n}\sin\alpha_n \end{pmatrix}.$$

(vii) Show that Ψ is a C^∞ diffeomorphism onto an open dense subset of B^{2n} with Jacobi determinant in absolute value equal to 2^{-n} and deduce $(\star\star)$.

Background. The preceding results imply that B^{2n} is diffeomorphic with the Cartesian product of n circles with a polytope of dimension n. Analogously, B^{2n+1} is diffeomorphic with the Cartesian product of n circles with the segment of the circular paraboloid of dimension $n+1$ given by

$$\{\,(y,z) \in \mathbf{R}_+^n \times \mathbf{R} \mid \sum_{1 \le j \le n} y_j + z^2 < 1\,\}$$

In v_n there occur as many factors π as there are independent ways to turn around in space, that is, the number of linearly independent (two-dimensional) planes. Phrased differently, the powers of π are given by the integral part of half the dimension.

(viii) According to the table above or the illustration below the sequence $(a_n)_{n=0}^{6}$ is strictly monotonically increasing while $a_6 > a_7 > a_8$. Combine these facts with $(\star)$ to prove that $(a_n)_{n=6}^{\infty}$ is strictly monotonically decreasing. Then apply part (vi) to show that $\lim_{n\to\infty} a_n = 0$. Deduce that also $(v_n)_{n=5}^{\infty}$ is strictly monotonically decreasing with $\lim_{n\to\infty} v_n = 0$.
Hint: One might use the following consequence of $(\star)$:

$$a_{n-1} = \frac{2\pi}{n-2}\frac{2\pi}{n-4}\cdots \begin{cases} \dfrac{2\pi}{7}a_6, & n \ge 7 \text{ odd;} \\ \dfrac{2\pi}{8}a_7, & n \ge 8 \text{ even.} \end{cases}$$

Accordingly, $a_6 = 33.073\cdots$ is the absolute maximum over all dimensions of the hyperareas of the corresponding unit spheres while $v_5 = 5.263\cdots$ is the absolute maximum over all dimensions of the volumes of the corresponding unit balls.

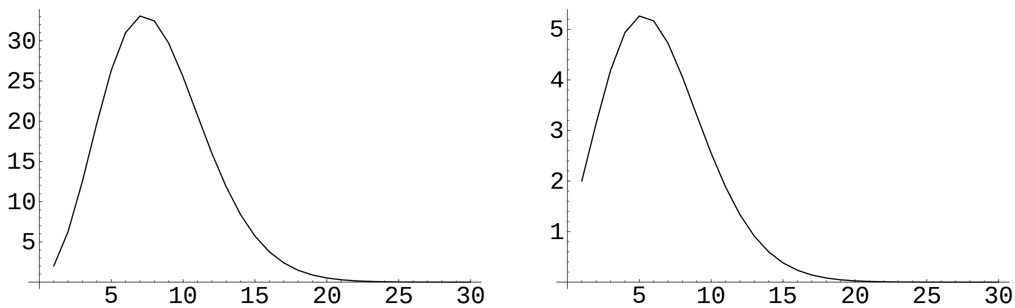

Illustration: Hyperarea a_{n-1} of unit sphere and volume v_n of unit ball, for $1 \le n \le 30$

Exercise 1.2 (Rate of change of circulation of vector field around moving curve). Write $I = [\,0,1\,]$, let $U \subset \mathbf{R}$ be open and suppose $\gamma : I^2 \to U$ is a C^1 mapping. Define the t-dependent compact curve

$$\gamma_t : I \to U \qquad \text{by} \qquad \gamma_t = \gamma(\cdot, t).$$

Let $f : U \to \mathbf{R}$ be a C^1 function. The rate of change of the integral of a function over a t-dependent curve is then given by the following formula, which is a direct consequence of the Fundamental Theorem 2.10.1 of Integral Calculus on $\mathbf{R}$:

$$\frac{d}{dt}\int_{\gamma_t(0)}^{\gamma_t(1)} f(x)\,dx = f_t(\gamma_t(1))\frac{\partial \gamma_t}{\partial t}(1) - f_t(\gamma_t(0))\frac{\partial \gamma_t}{\partial t}(0) \qquad (t \in I).$$

After this introductory remark we formulate an analogous result in dimension 3.

Let $U \subset \mathbf{R}^3$ be open and suppose $\gamma : I^2 \to U$ is a C^2 mapping. Define the t-dependent compact curve

$$\gamma_t : I \to U \qquad \text{by} \qquad \gamma_t = \gamma(\cdot, t), \qquad \text{and also} \qquad v \circ \gamma(s,t) := v_t \circ \gamma_t(s) := D_2\gamma(s,t) \in \mathbf{R}^3,$$

the velocity of the point $\gamma_t(s)$ at time $t \in I$. Let $f : U \to \mathbf{R}^3$ be a C^1 vector field on U. Consider

$$\int_{\gamma_t} \langle\, f(y), d_1 y\,\rangle = \int_I \langle\, f(\gamma(s,t),t),\, D_1\gamma(s,t)\,\rangle\, ds \qquad (t \in I),$$

the circulation of the vector field f around the curve γ_t. In two steps we will prove the following formula for the rate of change of this integral:

$$\frac{d}{dt}\int_{\gamma_t} \langle\, f(y), d_1 y\,\rangle = \int_{\gamma_t} \langle\, ((\mathrm{curl}\, f) \times v_t)(y), d_1 y\,\rangle + \langle\, f, v_t\,\rangle \circ \gamma_t(1) - \langle\, f, v_t\,\rangle \circ \gamma_t(0) \qquad (t \in I).$$

(i) Prove by means of the chain rule the following identities of functions on I^2:

$$D_2\langle\, f \circ \gamma,\, D_1\gamma\,\rangle - D_1\langle\, f \circ \gamma,\, D_2\gamma\,\rangle = \langle\, (Af) \circ \gamma \cdot D_2\gamma,\, D_1\gamma\,\rangle = \langle\, ((\mathrm{curl}\, f) \times v) \circ \gamma, D_1\gamma\,\rangle.$$

Here $Af(x) = Df(x) - Df(x)^t \in \mathrm{End}(\mathbf{R}^3)$ with ^{t} denoting the adjoint linear operator with respect to the standard inner product on $\mathbf{R}^3$.

(ii) Next, verify the formula for the rate of change on the basis of interchange of differentiation and integration and of part (i).