Solution of Exercise 1.1

(i) See Example 7.9.1.

(ii) $\phi(x, \alpha) = \phi(x', \alpha')$ implies by projection onto the first coordinate that $x = x'$. Consideration of the last two coordinates then leads to $\cos\alpha = \cos\alpha'$ and $\sin\alpha = \sin\alpha'$, that is $\alpha = \alpha'$. It is straightforward that $\mathrm{im}(\phi)$ is all of S^2 except the half-circle $\{\,(x, -s(x), 0) \in S^2 \mid |x| \le 1\,\}$ connecting the opposite points $x_\pm := (\pm 1, 0, 0)$. The half-circle is compact and of dimension 1 which implies that it is negligible for 2-dimensional integration (see page 526). We have

$$C^2 = \{\, x \in \mathbf{R}^3 \mid |x_1| < 1,\; x_2^2 + x_3^2 = 1\,\},$$

which shows that it is a cylinder, parallel to the x_1-axis. The preceding argument implies that ψ induces a bijection between C^2 and $S^2 \setminus \{x_\pm\}$. Given $(x, y) \in C^2$, its image $\psi(x, y) \in S^2$ may be obtained in the following geometrical manner. Denote by ℓ the unique straight line in $\mathbf{R}^3$ containing (x, y) that is parallel to the plane $\{\, x \in \mathbf{R}^3 \mid x_1 = 0\,\}$ and that intersects the x_1-axis. Next define $\psi(x, y)$ to be the point of intersection of ℓ with S^2 of shortest distance to (x, y).

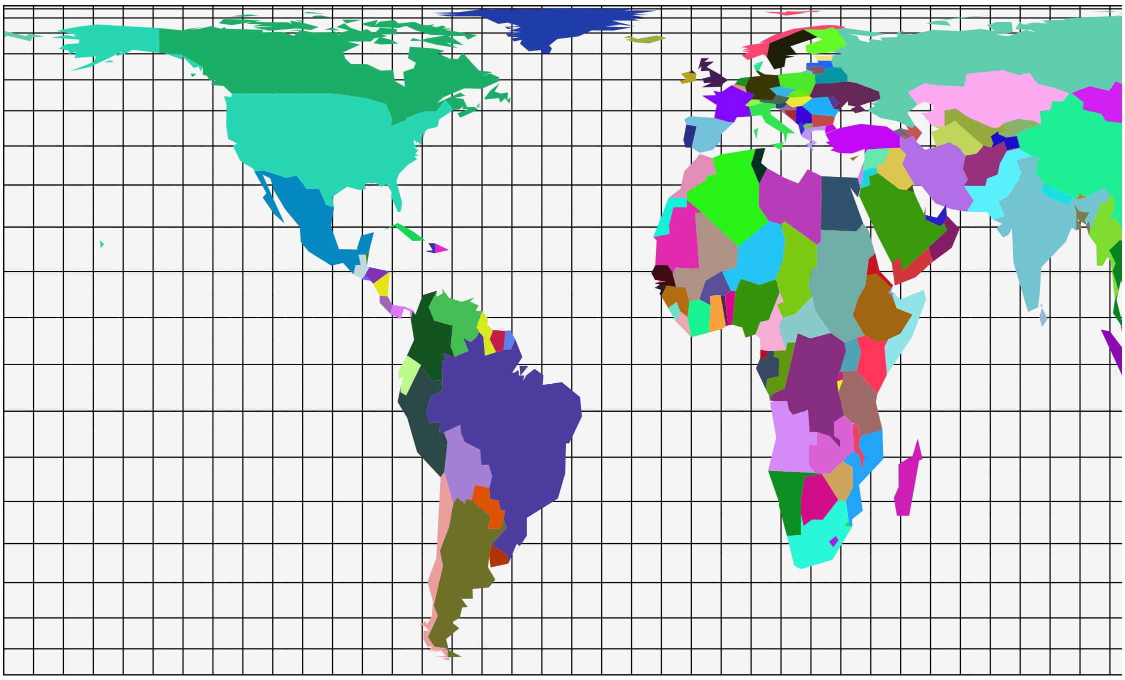

Illustration: Map of the surface of the Earth based on Lambert's cylindrical projection

(iii) On the basis of the chain rule one sees

$$D_j s(x) = \frac{1}{2s(x)}(-2x_j) = -\frac{x_j}{s(x)}; \qquad \text{in other words} \qquad \operatorname{grad} s(x) = -\frac{1}{s(x)} x^t,$$

which leads to the matrix for $D\phi(x, \alpha)$. Obviously $D\phi(x, \alpha)^t\, D\phi(x, \alpha)$ has the following matrix:

$$\begin{pmatrix} I_{n-2} & -\dfrac{\cos\alpha}{s(x)} x & -\dfrac{\sin\alpha}{s(x)} x \\ 0_{n-2} & -s(x)\sin\alpha & s(x)\cos\alpha \end{pmatrix} \begin{pmatrix} I_{n-2} & 0_{n-2} \\ -\dfrac{\cos\alpha}{s(x)} x^t & -s(x)\sin\alpha \\ -\dfrac{\sin\alpha}{s(x)} x^t & s(x)\cos\alpha \end{pmatrix}.$$

A-priori one knows the resulting matrix to be symmetric. Therefore, when multiplying the i-th row in the first matrix with the j-th column in the second, one has to distinguish only three cases: $1 \le i, j \le n-2$, which leads to the upper-left matrix belonging to $\mathrm{Mat}(n-2, \mathbf{R})$ in the answer; $i = j = n-1$, which gives the lower-right entry as a consequence of $\sin^2 + \cos^2 = 1$; and $i = n-1$ and $1 \le j \le n-2$, which leads to $\sin\alpha\cos\alpha\, x_j - \cos\alpha\sin\alpha\, x_j = 0$.

(iv) ϕ is of class C^∞ since all of its component functions are. Next $\mathrm{im}(\phi) \subset S^{n-1}$; indeed, for $(x, \alpha) \in D$,

$$\|\phi(x,\alpha)\|^2 = \|x\|^2 + s(x)^2(\cos^2\alpha + \sin^2\alpha) = \|x\|^2 + 1 - \|x\|^2 = 1.$$

Actually, $\mathrm{im}(\phi)$ is all of S^{n-1} except the set $\{\, (x, -s(x), 0) \in S^{n-1} \mid x \in \overline{B^{n-2}} \,\}$. This set is compact and of dimension $= \dim(B^{n-2}) = n-2$; that implies that it is negligible for $(n-1)$-dimensional integration (see page 526). Furthermore, ϕ is an embedding if it is immersive, injective and has a continuous inverse upon restriction to its image. Now, suppose $h \in \mathbf{R}^{n-1}$ satisfies $\mathbf{R}^n \ni D\phi(x,\alpha)h = 0$. In view of part (iii) the upper $n-2$ entries of the image vector give $h_1 = \cdots = h_{n-2} = 0$, while the two bottom entries lead to $(\sin^2\alpha + \cos^2\alpha)h_{n-1} = h_{n-1} = 0$. Accordingly, $D\phi(x,\alpha)$ is injective, for all $(x,\alpha) \in D$. As in part (ii) one shows directly that ϕ is injective on D. Finally, if $\phi(x,\alpha) = y \in \mathbf{R}^n$, then projection of y onto its upper $n-2$ entries produces x, while $\alpha = 2\arctan\left(\frac{y_n}{1+y_{n-1}}\right)$. This implies that the inverse mapping $\phi^{-1} : \phi(D) \to D$ with $\phi(x,\alpha) \mapsto (x,\alpha)$ is continuous.

(v) Exactly the same arguments as in the solution to Exercise 6.23.(iii) imply

$$\det\Big(I_{n-2} + \frac{1}{s(x)^2}xx^t\Big) = 1 + \frac{\|x\|^2}{s(x)^2} = \frac{1}{s(x)^2}.$$

As a consequence

$$\omega_\phi(x,\alpha) = \sqrt{\det\left(D\phi(x,\alpha)^t\, D\phi(x,\alpha)\right)} = \frac{1}{s(x)}s(x) = 1.$$

(vi) $\mathrm{im}(\phi) = S^{n-1}$ up to a negligible set according to part (iv), therefore one obtains from parts (v) and (i)

$$a_{n-1} = \int_{S^{n-1}} d_{n-1}y = \int_D \omega_\phi(y)\,dy = \int_{B^{n-2}} dx \int_{-\pi}^{\pi} d\alpha = 2\pi v_{n-2} = 2\pi\frac{a_{n-3}}{n-2}.$$

This implies directly

$$v_n = \frac{1}{n}a_{n-1} = \frac{2\pi}{n}\frac{a_{n-3}}{n-2} = \frac{2\pi}{n}v_{n-2}.$$

The formulae for v_n are a direct consequence of the identities $v_2 = \pi$ and $v_1 = 2$, while the formula for a_{2n-1} follows from part (i).

(vii) It is straightforward that Ψ is a C^∞ diffeomorphism onto its image. This image consists of B^{2n} under omission of the union of the origin and of all the sets (this union is negligible for $2n$-dimensional integration)

$$\{\, (x_1, \dots, x_{2j-1}, -z_j, 0, x_{2j+1}, \dots, x_{2n}) \in B^{2n} \mid 0 < z_j < 1 \,\} \qquad (1 \le j \le n).$$

Write $\Psi(y,\alpha) = \Psi'(y_1, \alpha_1, \cdots, y_n, \alpha_n)$. Since the difference between Ψ and Ψ' is a permutation of the coordinates, one has

$$|\det D\Psi(y,\alpha)| = |\det D\Psi'(y_1,\alpha_1,\cdots,y_n,\alpha_n)| = \prod_{1\le j\le n}\begin{vmatrix} \dfrac{\cos\alpha_j}{2\sqrt{y_j}} & -\sqrt{y_j}\sin\alpha_j \\ \dfrac{\sin\alpha_j}{2\sqrt{y_j}} & \sqrt{y_j}\cos\alpha_j \end{vmatrix} = \frac{1}{2^n}.$$