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Analyse in Meer Variabelen (WISB212)
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Exercise 0.1 (Laplacian of composition of norm and linear mapping). For x and $y \in \mathbb{R}^n$, recall that $\langle x, y\rangle = x^t y$ where x^t denotes the transpose of the column vector $x \in \mathbb{R}^n$; and furthermore, that $\|x\| = \sqrt{\langle x, x\rangle}$. Fix $A \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^p)$ and recall $\ker A = \{x \in \mathbb{R}^n | Ax = 0\}$. Now define $F : \mathbb{R}^n \ker A \to \mathbb{R}$ by $f = \|\cdot\| \circ A$, i.e. $f(x) = \|Ax\|$; and set $f^2(x) = f(x)^2$.

(i) Give an argument without computations that f is a positive C^∞ function.

(ii) By application of the chain rule to f^2 show, for $x \in \mathbb{R}^n \ker A$ and $h \in \mathbb{R}^n$,

$$Df(x)h = \frac{\langle Ax, Ah\rangle}{f(x)}.$$

Deduce that

$$Df(x) \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}) \qquad \text{is given by} \qquad Df(x) = \frac{1}{f(x)} x^t A^t A.$$

Denote by $(e_1, \cdots, e_n)$ the standard vectors in $\mathbb{R}^n$.

(iii) For $1 \le j \le n$, derive from part (ii) that

$$D_j f(x) = \frac{\langle Ax, Ae_j\rangle}{f(x)} \quad \text{and deduce} \quad D_j^2 f(x) = \frac{\|Ae_j\|^2}{f(x)} - \frac{\langle Ax, Ae_j\rangle^2}{f^3(x)}.$$

As usual, write $\triangle = \sum_{1\le j\le n} D_j^2$ for the Laplace operator acting in $\mathbb{R}^n$ and $\|A\|^2_{Eucl} = \sum_{1\le j\le n} \|Ae_j\|^2$.

(iv) Now demonstrate

$$\triangle(\|\cdot\| \circ A)(x) = \frac{\|A\|^2_{Eucl}\|Ax\|^2 - \|A^t Ax\|^2}{\|Ax\|^3}.$$

(v) Which form takes the preceding identity if A equasls the identity mapping in $\mathbb{R}^n$?

Exercise 0.2 (Application of Implicit Function Theorem). Suppose $d : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a C^∞ function and suppose there exists a C^∞ function $g : \mathbb{R} \to \mathbb{R}$ such that

$$G(0) \ne 0 \quad \text{and} \quad f(x; 0) = xg(x) \quad (x \in \mathbb{R}).$$

Consider the equation $f(x; y) = t$, where x and $t \in \mathbb{R}$, while $y \in \mathbb{R}^n$.

(i) Prove the existence of an open neighborhood V of 0 in $\mathbb{R}^n \times \mathbb{R}$ and of a unique C^∞ function $\psi : V \to \mathbb{R}$ such that, for all $(y, t) \in V$

$$\psi(0) = 0 \quad \text{and} \quad f(\psi(y, t); y) = t.$$

(ii) Establish the following formulae, where D_1 and D_2 denote differentiation with respect to the variables in $\mathbb{R}^n$ and $\mathbb{R}$, respectively:

$$D_1\psi(0) = -\frac{1}{g(0)} D_1 f(0; 0) \quad \text{and} \quad D_2\psi(0) = \frac{1}{g(0)}.$$

Exercise 0.3 (Quitic diffeomorphism). Recall that $\mathbb{R}_+ = \{x \in \mathbb{R} | x > 0\}$ and define

$$\Phi : \mathbb{R}^2_+ \to \mathbb{R}^2_+ \qquad \text{by} \qquad \Phi(x) = \frac{1}{(x_1 x_2)^2}(x_1^5, x_2^5).$$

(i) Prove that Φ is a C^∞ mapping and that $det D\Phi(x) = 5$, for all $x \in \mathbb{R}^2_+$.

(ii) Verify that Φ is a C^∞ diffeomorphism and that its inverse is given by

$$\Psi : \mathbb{R}^2_+ \to \mathbb{R}^2_+ \qquad \text{with} \qquad \Psi(y) = (y_1 y_2)^{\frac{2}{5}} (y_1^{\frac{1}{5}}, y_2^{\frac{1}{5}}).$$

Compute $det D\Psi(y)$, for all $y \in \mathbb{R}^2_+$.

Let $a > 0$ and define

$$g : \mathbb{R}^2 \to \mathbb{R} \qquad \text{by} \qquad g(x) = x_1^5 + x_2^5 - 5a(x_1 x_2)^2.$$

Now consider the bounded open sets

$$U = \{x \in \mathbb{R}^2_+ | g(x) < 0\} \qquad \text{and} \qquad V = \{y \in \mathbb{R}^2_+ | y_1 + y_2 < 5a\}.$$

Then U has a curved boundary, while V is an isosceles rectangular trinagle.

(iii) Show that $g \circ \Psi(y) = (y_1 y_2)^2 (y_1 + y_2 - 5a)$, for all $y \in \mathbb{R}^2_+$. Deduce that the restriction $\Psi|_V : V \to U$ is a diffeomorphism.

Background. By means of parts (ii) and (iii) one immediately computes the area of U to be $\frac{5a^2}{2}$.

Exercise 0.4 (Quintic analog of Descartes' folium). Let $g : \mathbb{R}^2 \to \mathbb{R}$ be the function from Exercise 0.3 and denote by F the zero-set of g (see the curve in the illustration above).

(i) Prove that F is a C^∞ submanifold in $\mathbb{R}^2$ of dimension 1 at every point of $F \backslash \{0\}$.

(ii) By means of intersection with lines through O obtain the following parametrization of a part of F:

$$\phi : \mathbb{R} \backslash \{-1\} \to \mathbb{R}^2 \quad \text{satisfying} \quad \phi(t) = \frac{5at^2}{1+t^5} \begin{pmatrix} 1 \\ t \end{pmatrix}.$$

(iii) Compute that

$$\phi'(t) = \frac{5at}{(1+t^5)^2} \begin{pmatrix} 2 - 3t^5 \\ t(3 - 2t^5) \end{pmatrix}.$$

Show that ϕ is an immersion except at 0.

(iv) Demonstrate that F is not a C^∞ submanifold in $\mathbb{R}^2$ of dimension 1 at 0.

The remainder is for extra credit and is no part of the regular exam. For $|x_2|$ small, x_2^5 is negligible; hence, after division by the common factor x_1^2 the equation $g(x) = 0$ takes the form $x_1^3 = 5a x_2^2$, which is the equation of an ordinary cusp. This suggest that F has a cusp at 0.

(v) Prove that F actually possesses two cusps at 0. This can be done with simple calculations; if necessary, however, one may use without proof

$$\begin{aligned} \phi''(t) &= \frac{10a}{(1+t^5)^3} \begin{pmatrix} 6t^{10} - 18t^5 + 1 \\ t(3t^{10} - 19t^5 + 3) \end{pmatrix}, \\ \phi'''(t) &= -\frac{30a}{(1+t^5)^4} \begin{pmatrix} 5t^4(2t^{10} - 16t^5 + 7) \\ 4t^5(t^{10} - 17t^5 + 13) - 1 \end{pmatrix}. \end{aligned}$$