TWEEDE DEELTENTAMEN WISB 212

Analyse in Meer Variabelen

03–07–2007   14–17 uur

- *Zet uw naam en collegekaartnummer op elk blad alsmede het totaal aantal ingeleverde bladzijden.*

- *De verschillende onderdelen van het vraagstuk zijn zoveel als mogelijk is, onafhankelijk van elkaar. Indien u een bepaald onderdeel niet of slechts ten dele kunt maken, mag u de resultaten daaruit gebruiken bij het maken van de volgende onderdelen. Raak dus niet ontmoedigd indien het u niet lukt een bepaald onderdeel te maken en ga gewoon door.*

- *Bij dit tentamen mogen boeken, syllabi, aantekeningen en/of rekenmachine **NIET** worden gebruikt.*

- *De twee vraagstukken tellen ieder voor de helft van het totaalcijfer.*

- *Het tentamen telt **VIER** bladzijden.*

Exercise 0.1 (Adjoints, vector calculus and quaternions). Write C for the linear space $C_c^\infty(\mathbf{R}^3)$ of C^∞ functions on $\mathbf{R}^3$ with compact support and introduce the usual inner product on C by $\langle f, g \rangle_C = \int_{\mathbf{R}^3} f(x)g(x)\, dx$, for f and $g \in C$. Consider the linear operator $D_j : C \to C$ of partial differentiation with respect to the j-th variable, for $1 \le j \le 3$.

(i) Prove that D_j is anti-adjoint with respect to the inner product on C, that is,

$$\langle D_j f, g \rangle_C = -\langle f, D_j g \rangle_C.$$

Denote by V the linear space of C^∞ vector fields on $\mathbf{R}^3$ with compact support and introduce an inner product on V by $\langle v, w \rangle_V = \int_{\mathbf{R}^3} \langle v(x), w(x) \rangle\, dx$, for v and $w \in V$. Here the inner product at the right-hand side is the usual inner product of vectors in $\mathbf{R}^3$. Furthermore, consider the linear operators $\operatorname{grad} : C \to V$ and $\operatorname{div} : V \to C$.

(ii) For $f \in C$ and $v \in V$, verify the following identity of functions in C:

$$\operatorname{div}(f\, v) = \langle \operatorname{grad} f,\, v \rangle + f \operatorname{div} v.$$

Use this to prove
$$\langle \operatorname{grad} f,\, v \rangle_V = -\langle f,\, \operatorname{div} v \rangle_C.$$
Conclude that $-\operatorname{div} : V \to C$ is the adjoint operator of $\operatorname{grad} : C \to V$.

(iii) For v and w in V, prove the following identity of functions in C:

$$\operatorname{div}(v \times w) = \langle \operatorname{curl} v,\, w \rangle - \langle v,\, \operatorname{curl} w \rangle.$$

Hint: At the left-hand side the operator D_1 only occurs in the term $D_1(v_2 w_3 - v_3 w_2)$ and apply Leibniz' rule. Next determine the occurrence of D_1 at the right-hand side.

(iv) Deduce from part (iii) that
$$\langle \operatorname{curl} v,\, w \rangle_V = \langle v,\, \operatorname{curl} w \rangle_V.$$
In other words, the linear operator $\operatorname{curl} : V \to V$ is self-adjoint.

Now consider the following matrix of differentiations acting on mappings $\left(\begin{smallmatrix} v \\ f \end{smallmatrix}\right) : \mathbf{R}^3 \to \mathbf{R}^4$:

$$M = \begin{pmatrix} \operatorname{curl} & \operatorname{grad} \\ -\operatorname{div} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -D_3 & D_2 & D_1 \\ D_3 & 0 & -D_1 & D_2 \\ -D_2 & D_1 & 0 & D_3 \\ -D_1 & -D_2 & -D_3 & 0 \end{pmatrix}.$$

The preceding results (in particular, part (i)) imply that M is a symmetric matrix, which **in this context** must be phrased as $M^t = -M$ (when "truly" transposing the matrix we also have to take the transpose of its coefficients).

(v) Verify that $-M^2$ equals Gram's matrix associated to M, that is, the matrix containing the inner products of the column vectors of M. Deduce $M^2 = -\Delta E$, where Δ is the Laplacian and E the 4×4 identity matrix. Derive, for $f \in C$ and $v \in V$

$$\operatorname{curl} \operatorname{grad} f = 0, \qquad \operatorname{div} \operatorname{curl} v = 0, \qquad \operatorname{curl}(\operatorname{curl} v) = \operatorname{grad}(\operatorname{div} v) - \Delta v,$$

where in the third identity the Laplacian Δ acts by components on v. Finally, show how to derive the second identity from the first.

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Background. We may write $M = D_1 I + D_2 J + D_3 K$, where I, J and $K \in \mathrm{Mat}(4, \mathbf{R})$ satisfy $I^2 = J^2 = K^2 = IJK = -E$. As a consequence $IJ = -JI = K$. Phrased differently, the linear space over $\mathbf{R}$ spanned by E, I, J, K provided with these rules of multiplication forms the noncommutative field $\mathbf{H}$ of the *quaternions*. In addition, analogously to the situation in dimension 1 where $(i\frac{d}{dx})^2 = -\frac{d^2}{dx^2}$, we have decomposed the Laplacian on $\mathbf{R}^3$ in a product of matrix-valued linear factors:

$$\Big(\frac{\partial}{\partial x_1}I + \frac{\partial}{\partial x_2}J + \frac{\partial}{\partial x_3}K\Big)^2 = -\Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\Big)E.$$

Exercise 0.2 (Left-invariant integration on $\mathrm{Mat}(n, \mathbf{R})$**).** As usual, we write $C_0(\mathbf{R}^n)$ for the linear space of continuous functions $f : \mathbf{R}^n \to \mathbf{R}$ having bounded support. Furthermore, we identify the linear space $\mathrm{Mat}(n, \mathbf{R})$ of $n \times n$ matrices over $\mathbf{R}$ with $\mathbf{R}^{n^2}$; in this way, by using n^2-dimensional integration, we assign a meaning to

$$\int_{\mathrm{Mat}(n,\mathbf{R})} f(X)\, dX \qquad \big(f \in C_0(\mathrm{Mat}(n, \mathbf{R}))\big).$$

(i) In particular, suppose $n = 2$ and consider the subgroup

$$\mathbf{SO}(2, \mathbf{R}) = \left\{ \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \in \mathrm{Mat}(2, \mathbf{R}) \,\middle|\, -\pi < \alpha \le \pi \right\}$$

of all orthogonal matrices in $\mathrm{Mat}(2, \mathbf{R})$ of determinant 1. Without proof one may use that ϕ is a C^∞ embedding if we define

$$\phi : \,]-\pi, \pi\,[\,\to \mathbf{R}^4 \qquad \text{by} \qquad \phi(\alpha) = (\cos\alpha, \sin\alpha, -\sin\alpha, \cos\alpha).$$

Now prove $\mathrm{vol}_1(\mathbf{SO}(2, \mathbf{R})) = 2\pi\sqrt{2}$.

(ii) Prove, for any $f \in C_0(\mathbf{R})$ with $0 \notin \mathrm{supp}\, f$ and any $0 \ne y \in \mathbf{R}$,

$$\int_{\mathbf{R}} \frac{f(y\,x)}{x}\, dx = \int_{\mathbf{R}} \frac{f(x)}{x}\, dx.$$

We now generalize the identity in part (ii) to $\mathrm{Mat}(n, \mathbf{R})$. We shall prove, for every $f \in C_0(\mathrm{Mat}(n, \mathbf{R}))$ with $\mathrm{supp}\, f \subset \mathbf{GL}(n, \mathbf{R})$ (= the group of invertible matrices in $\mathrm{Mat}(n, \mathbf{R})$) and $Y \in \mathbf{GL}(n, \mathbf{R})$,

$$(\star) \qquad \int_{\mathrm{Mat}(n,\mathbf{R})} \frac{f(YX)}{|\det X|^n}\, dX = \int_{\mathrm{Mat}(n,\mathbf{R})} \frac{f(X)}{|\det X|^n}\, dX.$$

Given $Y \in \mathbf{GL}(n, \mathbf{R})$, define

$$\Phi_Y : \mathrm{Mat}(n, \mathbf{R}) \to \mathrm{Mat}(n, \mathbf{R}) \qquad \text{by} \qquad \Phi_Y(X) = Y\,X.$$

(iii) Show that Φ_Y is a C^∞ diffeomorphism satisfying $D\Phi_Y(X) = \Phi_Y$, for all $X \in \mathrm{Mat}(n, \mathbf{R})$.

Denote by $e_1, \ldots, e_n$ the standard basis (column) vectors in $\mathbf{R}^n$, then a basis for $\mathrm{Mat}(n, \mathbf{R})$ is formed by the matrices

$$E_{i,j} = (0 \cdots 0 \; e_i \; 0 \cdots 0) \qquad (1 \le i,\, j \le n),$$

where e_i occurs in the j-th column. The ordering is lexicographic, but first with respect to j and then to i. In the case of $n = 2$ we thus obtain, in the following order:

$$E_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(iv) Verify $\Phi_Y(E_{i,j}) = (0 \ldots 0 \; Y e_i \; 0 \ldots 0)$. Deduce that the matrix of Φ_Y with respect to the $(E_{i,j})$ is given in block diagonal form with a copy of Y in each block and that $\det \Phi_Y = (\det Y)^n$.
Hint: First consider explicitly the case of $n = 2$, where the matrix of Φ_Y belongs to $\mathrm{Mat}(4, \mathbf{R})$. Then treat the general case.

(v) Prove $\Phi_Y(\mathbf{GL}(n, \mathbf{R})) \subset \mathbf{GL}(n, \mathbf{R})$. Now show the validity of $(\star)$ above by applying parts (iii) and (iv).

(vi) Select $Y \in \mathbf{GL}(n, \mathbf{R})$ satisfying $\det Y = -1$ and set $f(X) = \det X$. With these data $(\star)$ implies $-1 = 1$. Explain!