Department of Mathematics, Faculty of Science, UU.
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Inleiding Topologie Exam B (WISB-243)
29 Juni 2011

Excercise 1. Consider

$$\begin{aligned}
X_1 &= \left\{(x,\,y,\,z)\in\mathbb{R}^3\,:\,(z=0)\ \text{or}\ (x=y=0,\,z\geq 0)\right\}\\
X_2 &= \left\{(x,\,y,\,z)\in\mathbb{R}^3\,:\,(z=0)\ \text{or}\ \left(x=0,\,y^2+z^2=1,\,z\geq 0\right)\right\}\\
X_3 &= \left\{(x,\,y,\,z)\in\mathbb{R}^3\,:\,\left(x^2+y^2+z^2=1\right)\ \text{or}\ \left(y=0,\,z=0,\,\frac{1}{2}<|x|<1\right)\right\}
\end{aligned}$$

a) Show that X_1, X_2, X_3 are locally compact (hint: try to use the basic properties of locally compact spaces; alternatively, you can try to find direct arguments on the pictures).(0.5 punt)

b) Show that the one-point compactifications of X_1, X_2 and X_3 are homeomorphic to each other. (1 punt)

Excercise 2. Given a polynomial $p\in\mathbb{R}\left[X_0,\,X_1,\ldots,X_n\right]$, we denote $\mathcal{R}_p$ the set of reminders modulo p. In other words,

$$\mathcal{R}_p=\mathbb{R}\left[X_0,\,X_1,\ldots,X_n\right]/R_p,$$

Where R_p is the equivalence relation on $\mathbb{R}\left[X_0,\,X_1,\ldots,X_n\right]$ given by

$$R_p=\{(q_1,q_2)\,:\,\exists\,q\in\mathbb{R}\left[X_0,\,X_1,\ldots,X_n\right]\ \text{such that}\,q_1-q_2=pq\}$$

We also denote by $\pi_p\,:\,\mathbb{R}\left[X_0,\,X_1,\ldots,X_n\right]\longrightarrow\mathcal{R}_p$ the resulting quotient map. Show that

1. There is a unique algebra structure on $\mathcal{R}_p$ (i.e. unique operations $+$, $\cdot$ and multiplications by scalars, defines on $\mathcal{R}_p$) with the property that π_p is an morphism of algebras, i.e.

   $$\pi_p\left(q_1+q_2\right)=\pi\left(q_1\right)+\pi_2\left(q_2\right),\ \pi_p\left(q_1\cdot q_2\right)=\pi_p\left(q_1\right)\cdot\pi_p\left(q_2\right),\ \lambda\pi_p\left(q\right)=\pi_p\left(\lambda q\right)$$

   for all $q_1,\,q_2\in\mathbb{R}\left[X_0,\,X_1,\ldots,X_n\right],\ \lambda\in\mathbb{R}$. *(0.5 punt)*
2. For $p=x_0^2+\ldots+x_n^2$ the spectrum of $\mathcal{R}_p$ has only one point *(1 punt)*
3. For $p=x_0^2+\ldots+x_n^2-1$, the spectrum of $\mathcal{R}_p$ is homeomorphic to S^n *(1 punt)*
4. What is the spectrum for $p=x_0x_1\ldots x_n$? *(0.5 punt)*

Excercise 3. Let X be the space of continuous maps $f\,:\,[0,\,1]\longrightarrow[0,\,1]$ with the property that $f\left(0\right)=f\left(1\right)$. We endow it with the sup-metric d_{sup} and the induced topologie (recall that $d_{sup}\left(f,\,g\right)=sup\left\{\left|f\left(t\right)-g\left(t\right)\right|:t\in[0,\,1]\right\}$). Prove that:

1. X is not bounded and complete. *(1 punt)*
2. X is not compact. *(0.5 punt)*

Excercise 4. Show that:

1. The product of two sequentially compact spaces is sequentially compact. *(1 punt)*
2. Deduce that the product of two compact metric space is a compact space. *(1 punt)*

Excercise 5. Show that the family of open intervals

$$\mathcal{U} := \{(q, q+1) : q \in \mathbb{R}\}$$

forms an open cover of $\mathbb{R}$ which admits no finite sub-cover, but which admits a locally finite sub-cover. *(1.5p)*

Excercise 6. Prove that there is no continuous injective map $f : S^1 \vee S^1 \longrightarrow S^1$, where $S^1 \vee S^1$ is a bouquet of two circles (two copies of S^1, tangent to each other). *(1.5 punt)*

Note: the mark for this exam is the minimum between 10 and the number of points that you score (in total, there are 11 points in the game!).