Inleiding Topologie (Retake, March 12, 2014)

Exercise 1. Let $\mathcal{T}$ be the smallest topology on $\mathbb{R}$ with the property that

$$f : (\mathbb{R}, \mathcal{T}) \longrightarrow (\mathbb{R}, \mathcal{T}_{\text{Eucl}}), \quad f(x) = x^2$$

is continuous.

a. Describe a basis of $(\mathbb{R}, \mathcal{T})$ and show that any $U \in \mathcal{T}$ has the property that

$$-x \in U \quad \forall \ x \in U.$$

b. Find the closure and the interior of $(-1, 2)$ in $(\mathbb{R}, \mathcal{T})$.

c. Is $(\mathbb{R}, \mathcal{T})$ Hausdorff? Can you find a sequence with two (distinct) limits?

d. Is $[-1, 1]$ (with the topology induced from $\mathcal{T}$) compact? But connected?

e. Is $[-1, 1)$ (with the topology induced from $\mathcal{T}$) compact?

f. Is $[-3, 1) \cup (1, 3]$ (with the topology induced from $\mathcal{T}$) connected?

g. Does there exist a metric space X and a finite group Γ acting on X such that X/Γ is homeomorphic to $(\mathbb{R}, \mathcal{T})$?

Exercise 2. Let $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ be a continuous function between two *Hausdorff* topological spaces. Define $f^*\mathcal{T}_Y$ as the smallest topology on X with the property that

$$f : (X, f^*\mathcal{T}_Y) \longrightarrow (Y, \mathcal{T}_Y)$$

is continuous, and define $f_*\mathcal{T}_X$ as the largest topology on Y with the property that

$$f : (X, \mathcal{T}_X) \longrightarrow (Y, f_*\mathcal{T}_X)$$

is continuous. Show that

a. $f^*\mathcal{T}_Y = \{f^{-1}(V) : V \in \mathcal{T}_Y\}$ and $f_*\mathcal{T}_X = \{V \subset Y : f^{-1}(V) \in \mathcal{T}_X\}$.

b. If f is a homeomorphism then $f_*\mathcal{T}_X = \mathcal{T}_Y$ and $f^*\mathcal{T}_Y = \mathcal{T}_X$.

c. If $f_*\mathcal{T}_X = \mathcal{T}_Y$ and $(Y, \mathcal{T}_Y)$ is connected, then f is surjective.

d. If $f^*\mathcal{T}_Y = \mathcal{T}_X$ then f is injective.

e. $f^*\mathcal{T}_Y = \mathcal{T}_X$ holds if and only if f is an embedding.

Exercise 3. Assume that X is a sphere S^2 minus n points and Y is a sphere S^2 minus m points, where $m, n \geq 0$ are integers. Show that if X is homeomorphic to Y, then $n = m$.

Exercise 4. Assume that $(X, \mathcal{T}_X)$ is a compact Hausdorff space and A is a closed subset of X. We consider the complement of A in X,

$$Y := X - A = \{x \in X : x \notin A\},$$

we denote by X/A the space obtained from X by collapsing A to a point and we consider the canonical projection:

$$\pi : X \longrightarrow X/A$$

(recall that X/A is endowed with the topology $\pi_* \mathcal{T}_X$). Show that:

a. Y is locally compact and Hausdorff.

b. For any open U in X with the property that $U \cap A = \emptyset$ or $A \subset U$, one has that $\pi(U)$ is open in X/A.

c. X/A is a compact Hausdorff space.

d. The one point compactification of Y is homeomorphic to X/A.

Exercise 5. Let $\mathcal{C}(D^2)$ be the algebra of real-valued continuous functions on the unit disk D^2 and let $\mathcal{A}$ be the subset consisting of those $f \in \mathcal{C}(D^2)$ with the property that they are constant on the boundary circle S^1.

a. is $\mathcal{A}$ a sub-algebra of $\mathcal{C}(D^2)$?

b. is $\mathcal{A}$ dense in $\mathcal{C}(D^2)$?

c. show that the spectrum of $\mathcal{A}$ is homeomorphic to S^2.

Notes:

1. You may give your answers in Dutch or English.
2. All the questions a., b. etc are worth 0.5 points. Exercise 3 is worth 1.5 points (note also that the sum of all the points is 11 ...).
3. As before, you are allowed to use during the exam the three sheets of A4 papers (= six pages) containing definitions, theorems, etc from the course- that you prepared at home.
4. PLEASE MOTIVATE ALL YOUR ANSWERS!!!!!!!!!! (give details, explain your reasoning, use pictures whenever appropriate, etc etc).