Inleiding Topologie (Retake, March 11, 2015)

Note: Please explain/prove your answers. E.g., in Exercise 1, parts c., d., f. please do not just say "yes" or "no" but also prove your claims. Also, in part e. or h., do not just write down the final result, but also explain how you found it. Also, in Exercise 2, do not forget to prove that the examples you give are not homeomorphic; etc. etc.

Exercise 1. Let k be a natural number ($k \in \{0, 1, 2, \ldots\}$). Let $\mathcal{B}_k$ be the family of subsets of $\mathbb{R}$ consisting of $\mathbb{R}$ and all the open intervals (a, b) which contain at most k integers (when $k = 0$, that means: intervals (a, b) that contain no integers). Show that:

a. $\mathcal{B}_k$ is a topology basis on $\mathbb{R}$. *(0.5 points)*

b. for $k \geq 1$, the topology induced by $\mathcal{B}_k$ on $\mathbb{R}$ is the Euclidean topology. *(0.5 points)*

Next, we assume that $k = 0$ and we look at the topology $\mathcal{T}_0 := \mathcal{T}(\mathcal{B}_0)$ induced by $\mathcal{B}_0$.

c. Is $(\mathbb{R}, \mathcal{T}_0)$ Hausdorff? *(0.5 points)*

d. Is $(\mathbb{R}, \mathcal{T}_0)$ compact? *(0.5 points)*

e. Compute the interior and the closure of $A = [0, 2)$ in $(\mathbb{R}, \mathcal{T}_0)$. *(0.5 points)*

f. Is the sequence $(x_n)_{n \geq 1}$ given by

$$x_n = \frac{n^{2015} + sin(n)}{n^2}$$

convergent in $(\mathbb{R}, \mathcal{T}_0)$? *(0.5 points)*

g. Show that any continuous function $f : (\mathbb{R}, \mathcal{T}_0) \to (\mathbb{R}, \mathcal{T}_{Eucl})$ is constant. *(0.5 points)*

h. Compute the interior and the closure of

$$D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$$

in $\mathbb{R}^2$ endowed with the product topology $\mathcal{T}_0 \times \mathcal{T}_{Eucl}$. *(1 point)*

Exercise 2. Find two locally compact Hausdorff spaces X and Y which are not homeomorphic but which have homeomorphic one-point compactifications. *(0.75 pts)* Can you find such X and Y such that all the spaces X, Y, X^+ and Y^+ are connected? *(0.75 pts)*

Exercise 3. Let $\Gamma = \mathbb{R}_{>0}$ be the group of strictly positive real numbers, endowed with the usual multiplication. Let $X = \mathbb{R}^n \setminus \{0\}$. Show that

$$\Gamma \times X \to X, \quad (r, x) \mapsto rx$$

(the usual multiplication of vectors x by scalars r) defines an action of Γ on X and prove that X/Γ is homeomorphic to S^{n-1} (the $n-1$-dimensional sphere). *(1 point)*

Exercise 4. Let X be a topological space.

a. If $\gamma_1, \gamma_2 : [0,1] \to X$ are continuous and $\gamma_1(1) = \gamma_2(0)$, show that

$$\gamma : [0,1] \to X, \quad \gamma(t) = \begin{cases} \gamma_1(2t) & \text{if } t \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & \text{if } t \in (\frac{1}{2}, 1] \end{cases}$$

is continuous. *(0.5 points)*

b. If there exist $A, B \subset X$ path connected such that $X = A \cup B$ and $A \cap B \neq \emptyset$, then X is connected. *(0.5 points)*

Exercise 5. Let A be an algebra over $\mathbb{R}$, assume that $\chi : A \to \mathbb{R}$ is a character and let

$$I := \mathrm{Ker}(\chi) \quad (= \{a \in A : \chi(a) = 0\}).$$

Show that:

a. I is an ideal of A. *(0.5 points)*

b. $a - \chi(a) \cdot \mathbb{1}_A \in I$ for all $a \in A$ (where $\mathbb{1}_A$ is the unit of A). *(0.5 points)*

c. I is a maximal ideal. *(0.5 points)*

Exercise 6. Let X be a compact Hausdorff space,

$$F = (F_1, \ldots, F_n) : X \to \mathbb{R}^n$$

a continuous function and consider

$$\mathcal{A}_F = \{f \in C(X) : f = P \circ F \ \text{ for some polynomial function } P : \mathbb{R}^n \to \mathbb{R}\}.$$

Show that F is an embedding if and only if $\mathcal{A}_F$ is dense in $C(X)$. *(1.5 points)*
(Recall: $C(X)$ denotes the space of all continuous functions from X to $\mathbb{R}$, and it is a topological space with the topology induced by the sup-norm).