Topology and Geometry A (Inleiding Topologie), Retake (August 27, 2010)

Note: Please motivate/prove each of your answers.

Exercise 1. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of non-negative integers. We consider the following two collections of subsets of $\mathbb{N}$:

- $\mathcal{T}_1$ consisting of $\emptyset$, $\mathbb{N}$ and all the sets of the form $\{0, 1, \ldots, n\}$ with $n \in \mathbb{N}$.
- $\mathcal{T}_2$ consisting of $\emptyset$ and all the sets of the form $\{n, n+1, \ldots\}$ with $n \in \mathbb{N}$.

Questions:

(1) Show that $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on $\mathbb{N}$. (*1p*)

(2) Show that the spaces $(\mathbb{N}, \mathcal{T}_1)$ and $(\mathbb{N}, \mathcal{T}_2)$ are not homeomorphic. (*0.5p*)

(3) For each of the spaces $(\mathbb{N}, \mathcal{T}_1)$ and $(\mathbb{N}, \mathcal{T}_2)$ decide whether the space is Hausdorff, connected or compact. (*1.5p*)

Exercise 2. Let $\mathcal{T}_u$ be the topology on $\mathbb{R}$ induced by the topology basis:

$$\mathcal{B}_u := \{[(a, b] : a, b \in \mathbb{R}, a < b\}.$$

Compute the interior, the closure and the boundary of

$$A := ((0, \frac{1}{3}] \cup [\frac{1}{2}, 1]) \times [0, 1)$$

in the topological space $X = \mathbb{R} \times \mathbb{R}$ endowed with the product topology $\mathcal{T}_u \times \mathcal{T}_u$. (*2.5p*)

Exercise 3. Let X be the connected sum of a Moebius band and a torus (Figure 1). Show how one can obtain X from a disk by gluing some of the points on the boundary of the disk. (*2p*) Then describe on the picture a continuous map $f : S^1 \longrightarrow X$ such that the one-point compactification of X is homeomorphic to a sphere. (*0.5p*)

Exercise 4. Let X be the one-point compactification of the space obtained by removing two points from the torus. Show that:

1. X can be embedded in $\mathbb{R}^3$. (*1p*)
2. X is not homeomorphic to S^2. (*1p*)

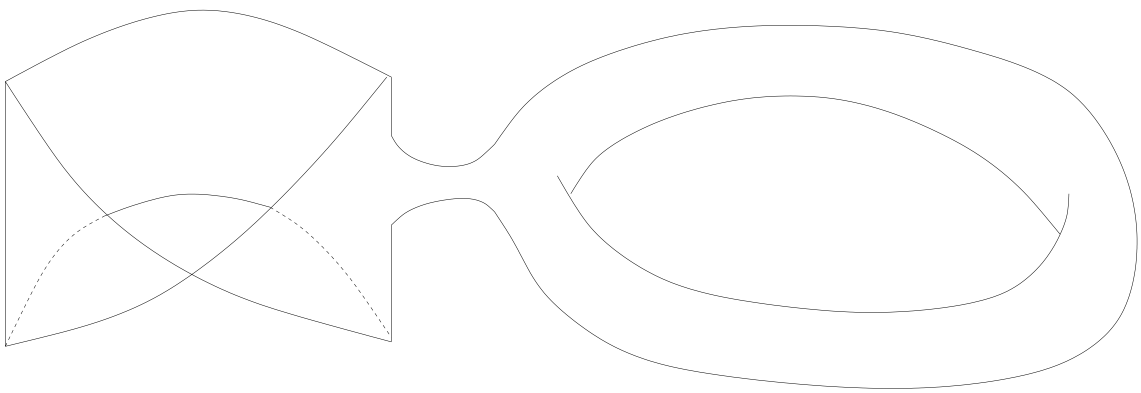

Figure 1: