Inleiding Topologie – Exam 1 – April 20, 2011

Name: ______________________________

Notes:

(a) justify all your answers!!!

(b) the marking starts from 1 point. By solving the exercises, you can earn 9.5 more points. Your mark for this exam will be the minimum between your total number of points and 10.

1. On $X = \mathbb{R}$ consider the family of subsets:

$$\mathcal{B} := \{(-p, p) : p \in \mathbb{Q}, p > 0\}, \ \mathcal{T} = \{(-a, a) : 0 \leq a \leq \infty\}.$$

   (a) Show that $\mathcal{B}$ is a topology basis (*0.5 pt*).

   (b) Show that $\mathcal{T}$ is the topology associated to $\mathcal{B}$. (*0.5 pt*).

   (c) Is the sequence $x_n = (-1)^n + \frac{1}{n}$ convergent in $(X, \mathcal{T})$? To what? (*0.5 pt*).

   (d) Find the interior and the closure of $A = (-1, 2)$ in $(X, \mathcal{T})$ (*0.5 pt*).

   (e) Show that any continuous function $f : X \to \mathbb{R}$ is constant (*0.5 pt*).

   (f) For the topological space $(X, \mathcal{T})$, decide whether it is:

      1. Hausdorff (*0.5 pt*).
      2. 1st countable (*0.5 pt*).
      3. Metrizable (*0.5 pt*).
      4. Connected (*0.5 pt*).

2. Which of the following spaces are homeomorphic and which are not:

   (a) $(1, \infty)$ and $(0, \infty)$ (*0.5 pt*).

   (b) $\mathbb{R}^2 - D^2$ and $\mathbb{R}^2 - \{0\}$ (*0.5 pt*).

   (c) $(0, 1)$ and $[0, 1)$ (*0.5 pt*).

   (d) $S^1 \times (\mathbb{R}^2 - \{0\})$ and $T^2 \times \mathbb{R}^*$ (*0.5 pt*).

   (e) $S^1 \times (\mathbb{R}^2 - \{0\})$ and $T^2 \times \mathbb{R}^*_+$ (*0.5 pt*).

   (here D^2 denotes the closed unit disk, T^2 denoted the torus. $\mathbb{R}^* = \mathbb{R} - \{0\}$, $\mathbb{R}^*_+ = (0, \infty)$).

3. Consider the map

$$\pi\colon \mathbb{R}^2 \to S^1 \times \mathbb{R}, \qquad (x,y) \mapsto \Big(\big(\cos(x+y), \sin(x+y)\big),\, x-y \Big) \in S^1 \times \mathbb{R}.$$

   (a) Describe an equivalence relation R on $\mathbb{R}^2$ such that $(S^1 \times \mathbb{R}, \pi)$ is a quotient of $\mathbb{R}^2$ modulo R (*0.5 pt*).

   (b) Find a group Γ and an action of Γ on $\mathbb{R}^2$ such that R is the equivalence relation induced by this action (*0.5 pt*).

   (c) Show that, indeed, $\mathbb{R}^2/\Gamma$ is homeomorphic to $S^1 \times \mathbb{R}$ (*0.5 pt*).

4. Show that, if a topological space X is Hausdorff, then the cone $\mathrm{Cone}(X)$ of X is Hausdorff (*0.5 pt*).

5. Show that any continuous function $f : [0,1] \to [0,1]$ admits a fixed-point, i.e. there exists $t_0 \in [0,1]$ such that $f(t_0) = t_0$. (*0.5 pt*).