Inleiding Topologie, Exam B (June 27, 2012)

Exercise 1. (*1p*) Show that

$$K := \{(x,y) \in \mathbb{R}^2 : x^{2012} + y^{2012} \leq 10sin(e^x + e^y + 1000) + e^{cos(x^2+y^2)}\}.$$

is compact.

Exercise 2. (*1.5 p*) Let X be a bouquet of two circles:

$$X = \{(x,y) \in \mathbb{R}^2 : ((x-1)^2 + y^2 - 1)((x+1)^2 + y^2 - 1) = 0\}.$$

We say that a space Y is an *exam-space* if there exist three distinct points $p, q, r \in X$ such that Y is homeomorphic to the one point compactification of $X - \{p, q, r\}$.

Find the largest number l with the property that there exist exam-spaces $Y_1, \ldots, Y_l$ with the property that any two of them are not homeomorphic (prove all the statements that you make!).

Exercise 3. (*1p*) Let X be a topological space and let $\gamma : [0,1] \longrightarrow X$ be a continuous function. Assume that γ is locally injective i.e., for any $t \in [0,1]$, there exists a neighborhood V of t in $[0,1]$ such that

$$\gamma|_V : V \longrightarrow X$$

is injective. Show that, for any $x \in X$, the set

$$\gamma^{-1}(x) := \{t \in [0,1] : \gamma(t) = x\}$$

is finite.

Exercise 4. (*1p*) Let X be a normal space and let $A \subset X$ be a subspace with the property that any two continuous functions $f, g : X \longrightarrow \mathbb{R}$ which coincide on A must coincide everywhere on X. Show that A is dense in X (i.e. the closure of A in X coincides with X).

Exercise 5. (*1p*) Consider the following open cover of $\mathbb{R}$:

$$\mathcal{U} := \{(r,s) : r, s \in \mathbb{R}, |r - s| < \frac{1}{3}\}.$$

Describe a locally finite subcover of $\mathcal{U}$.

Exercise 6. (each of the sub-questions is worth *0.5 p*) Let A be a commutative algebra over $\mathbb{R}$. Assume that it is finitely generated, i.e. there exist $a_1, \ldots, a_n \in A$ (called generators) such that any $a \in A$ can be written as

$$a = P(a_1, \ldots, a_n),$$

for some polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$. Recall that X_A denotes the topological spectrum of A; consider the functions

$$f_i : X_A \longrightarrow \mathbb{R}, \quad f_i(\chi) = \chi(a_i) \quad 1 \leq i \leq n,$$
$$f = (f_1, \ldots, f_n) : X_A \longrightarrow \mathbb{R}^n.$$

Show that

(i) f is continuous.

(ii) For any character $\chi \in X_A$ and any polynomial $P \in \mathbb{R}[X_1, \ldots, X_n]$,

$$\chi(P(a_1, \ldots, a_n)) = P(\chi(a_1), \ldots, \chi(a_n)).$$

(iii) f is injective.

(iv) the topology of X_A is the smallest topology on X_A with the property that all the functions f_i are continuous.

(v) f is an embedding.

Next, for a subspace $K \subset \mathbb{R}^n$, we denote by $\mathrm{Pol}(K)$ the algebra of real-valued polynomial functions on K and let $a_1, \ldots, a_n \in \mathrm{Pol}(K)$ be given by

$$a_i : K \longrightarrow \mathbb{R}, \quad a_i(x_1, \ldots, x_n) = x_i.$$

Show that

(vi) $\mathrm{Pol}(K)$ is finitely generated with generators $a_1, \ldots, a_n$.

(vii) Show that the image of f (from the previous part) contains K.

Finally:

(viii) For the $(n-1)$ sphere $K = S^{n-1} \subset \mathbb{R}^n$, deduce that f induces a homeomorphism between the spectrum of the algebra $\mathrm{Pol}(K)$ and K.

(ix) For which subspaces $K \subset \mathbb{R}^n$ can one use a similar argument to deduce that the spectrum of $\mathrm{Pol}(K)$ is homeomorphic to K?

Note: Motivate all your answers; give all details; please write clearly (English or Dutch). The mark is given by the formula:

$$\mathbf{min}\{10, 1 + p\},$$

where p is the number of points you collect from the exercises.