Herkansing, Inleiding Topologie, August 22, 2012

Exercise 1 On $X = \mathbb{R}$ consider the topology:

$$\mathcal{T} = \{(-a, a) : 0 \leq a \leq \infty\}.$$

(i) Is $(X, \mathcal{T})$ metrizable? (*0.5 p*)

(ii) Is $(X, \mathcal{T})$ 1st countable? (*0.5 p*)

(iii) Is $(X, \mathcal{T})$ connected? (*0.5 p*)

(iv) Is the sequence $x_n = (-1)^n + \frac{1}{n}$ convergent in $(X, \mathcal{T})$? To what? (*0.5 p*)

(v) Find the interior and the closure of $A = (-1, 2)$ in $(X, \mathcal{T})$. (*0.5 p*)

(vi) Show that any continuous function $f : X \to \mathbb{R}$ is constant. (*0.5 p*)

Exercise 2 Let M be the Moebius band. For a continuous map $f : S^1 \longrightarrow M$ we denote

$$X_f := M - f(S^1).$$

(i) Show that, for any f, X_f is locally compact but not compact. (*0.5 p*)

(ii) Is there a function f such that X_f^+ is homeomorphic to the projective space $\mathbb{P}^2$? (*0.5 p*)

Exercise 3

(i) Let A be a commutative algebra over $\mathbb{R}$ and assume that $a_0, a_1, \ldots, a_n \in A$ generate A, i.e. that any $a \in A$ can be written as

$$a = P(a_0, \ldots, a_n),$$

for some $P \in \mathbb{R}[X_0, \ldots, X_n]$. Let X_A be the topological spectrum of A. Show that

$$f : X_A \longrightarrow \mathbb{R}^{n+1}, \quad f(\chi) = (\chi(a_0), \ldots, \chi(a_n))$$

is an embedding. (*1.5 p*)

(ii) If $A = \mathrm{Pol}(K)$ is the algebra of real-valued polynomial functions on a subset $K \subset \mathbb{R}^{n+1}$ and a_i are the polynomial functions

$$a_i : K \longrightarrow \mathbb{R}, \quad a_i(x_1, \ldots, x_n) = x_i, \quad (0 \leq i \leq n),$$

show that the image of the resulting f contains K. (*0.5 p*)

(iii) For the sphere $S^n \subset \mathbb{R}^{n+1}$, deduce that the spectrum of the algebra $\mathrm{Pol}(S^n)$ is homeomorphic to S^n. (*1 p*)

Exercise 4 *(1 p)* Prove that there is no continuous map $g : \mathbb{C} \longrightarrow \mathbb{C}$ with the property that $g(z)^2 = z$ for all $z \in \mathbb{C}$.

Exercise 5 *(1 p)* On $(0, \infty)$ we define the action of the group $(\mathbb{Z}, +)$ by:

$$\mathbb{Z} \times (0, \infty) \longrightarrow (0, \infty), \quad (n, r) \mapsto 2^n r.$$

Show that the quotient $(0, \infty)/\mathbb{Z}$ is homeomorphic to S^1.

Exercise 6 *(1 p)* Let X be a normal space. Show that, for $A, B \subset X$, A and B have the same closure in X if and only if, for any continuous function $f : X \longrightarrow \mathbb{R}$, one has the equivalence

$$f|_A = 0 \Longleftrightarrow f|_B = 0.$$

Note: Please motivate all your answers.