Exam Inleiding Topologie, 30/1-2017, 13:30 - 16:30

Solution 1.

(a) Let $a < b, a' < b'$ and $x \in \mathbb{R}$ be real numbers such that $x \in [a,b) \cap [a',b')$. Then $a'' := \max(a,a') \leq x$ and $b'' := \min(b,b') > x$. It follows that $x \in [a'',b'') \subset [a,b) \cap [a',b')$. This establishes the assertion.

(b) It is straightfoward to see that T is a bijection with inverse $T^{-1} : y \mapsto p^{-1}y - q/p$. Thus we see that the pre-image of an interval of the form $[a,b)$ equals

$$T^{-1}([a,b)) = [a',b'),$$

with $a' = a/p - q/p$ and $b' = b/p - q/p$. Thus, $T^{-1}([a,b)) \in \mathscr{T}_l$. Since the sets $[a,b)$ form a basis of $\mathscr{T}_l$ we see that T is continuous. Since T^{-1} is of similar type, we see that T^{-1} is continuous as well. Hence, T is a homeomorphism.

(c) We first observe that

$$(0,1) = \bigcup_{n \geq 1} [\frac{1}{n}, 1).$$

Thus, $(0,1)$ is a union of sets from $\mathscr{T}_l$. By applying item (b), we find that every set of the form $(q, q+p)$ with $p, q \in \mathbb{R}$ and $p > 0$ belongs to $\mathscr{T}_l$. Since the sets $(q, q+p)$ form a basis of the topology for $\mathscr{T}_{\text{eucl}}$, the inclusion follows.

(d) Let $x, y \in \mathbb{R}$, $x \neq y$. Since $(\mathbb{R}, \mathscr{T}_{\text{eucl}})$ is (metrizable hence) Hausdorff, there exist $U, V \in \mathscr{T}_{\text{eucl}}$ such that $U \ni x$, $V \ni y$ and $U \cap V = \emptyset$. By (c) we have $U, V \in \mathscr{T}_l$. Hence, $(\mathbb{R}, \mathscr{T}_l)$ is Hausdorff.

(e) The identity map $I : \mathbb{R} \to \mathbb{R}$ is continuous $(\mathbb{R}, \mathscr{T}_l) \to (\mathbb{R}, \mathscr{T}_{\text{eucl}})$ and maps S to S. Thus, if S is compact in $(\mathbb{R}, \mathscr{T}_l)$ then its image S under I is compact in $(\mathbb{R}, \mathscr{T}_{\text{eucl}})$.

Alternative solution: Assume that S is compact with respect to $\mathscr{T}_l$. Let $\{U_i\}_{i \in I}$ be an open cover of S with sets from $\mathscr{T}_{\text{eucl}}$. By the previous item, each set U_i belongs to $\mathscr{T}_l$, so that $\{U_i\}_{i \in I}$ is an open cover of S relative to $\mathscr{T}_l$. Since S is compact relative to $\mathscr{T}_l$ the cover contains a finite subcover. Hence, S is compact relative to $\mathscr{T}_{\text{eucl}}$.

(f) We observe that $[a, \infty) = \cup_{n>1} [a, n)$ belongs to $\mathscr{T}_l$ hence its complement $(-\infty, a)$ is closed in $(\mathbb{R}, \mathscr{T}_l)$ and it follows that $S \cap (-\infty, a)$ is closed in S, relative to (the restriction of) $\mathscr{T}_l$. Since S is compact for $\mathscr{T}_l$, it follows that $S \cap (-\infty, a)$ is compact for $\mathscr{T}_l$.

(g) The set $[0,1) = [0,1] \cap (-\infty, 1)$ is closed in $[0,1]$, relative to the topology induced by $\mathscr{T}_l$, by item (f). If $[0,1]$ were compact for $\mathscr{T}_l$, then $[0,1) = [0,1] \cap (-\infty, 1)$ would be compact for $\mathscr{T}_l$ by hence also for $\mathscr{T}_{\text{eucl}}$, by (e). This is a contradiction, since all compact subsets of $(\mathbb{R}, \mathscr{T}_{\text{eucl}})$ are closed in $(\mathbb{R}, \mathscr{T}_{\text{eucl}})$. It follows that $[0,1]$ is not compact for $\mathscr{T}_l$.

(h) Assume $(\mathbb{R}, \mathscr{T}_l)$ were locally compact. Then there would be a compact neighborhood N of 0 relative to $\mathscr{T}_l$. Now N would contain a set of the form $[0, 2\delta) \in \mathscr{T}_l$, for $\delta > 0$. Hence $N \supset [0, \delta]$. The set $[0, \delta]$ is closed in $(\mathbb{R}, \mathscr{T}_{\text{eucl}})$ hence in $(\mathbb{R}, \mathscr{T}_l)$, by (c). It follows that $[0, \delta]$ is closed in N relative to the restriction of $\mathscr{T}_l$, hence compact. This contradicts the conclusion of the previous item, in view of (b).

Solution 2.

(a) By definition, Y is the collection of sets Γx, for $x \in \mathbb{R}$. Furthermore, $\pi : \mathbb{R} \to Y$ is given by $\pi(x) = \Gamma x$. Now $\Gamma \cdot 0 = \{0\}$, $\Gamma \cdot (-1) = (-\infty, 0)$ and $\Gamma \cdot 1 = (0, \infty)$. The unit of these sets is $\mathbb{R}$. Thus, we see that $\mathbb{R}$ splits into 3 Γ-orbits, namely the ones containing $-1, 0, 1$. These orbits are precisely the points a, b and c in Y.

(b) A set $S \subset Y$ is open for the quotient topology if and only if $\pi^{-1}(S)$ is open. Now $\pi(S)$ is the union of the fibers $\pi^{-1}(y)$, for $y \in Y$. The fibers are: $\pi^{-1}(a) = \Gamma \cdot (-1) = (-\infty, 0)$ $\pi^{-1}(b) = \Gamma \cdot 0 = \{0\}$ and $\pi^{-1}(c) = \Gamma \cdot 1 = (0, \infty)$. From this we see that

$$\mathscr{T}_Y \supset \{\emptyset,\ Y,\ \{a\},\ \{c\}, \{a, c\}\}.$$

If $U \in \mathscr{T}_Y$ contains b, then $\pi^{-1}(U)$ must contain 0. For it to be a union of the fibers and open in $\mathbb{R}$, it needs to contain $\mathbb{R}$. Hence, $U = Y$. It follows that the inclusion $\supset$ is an equality.

(c) The space Y is not Hausdorff. Indeed, the only set from $\mathscr{T}_Y$ containing b is Y. Thus, every neighborhood of b contains Y and we see that this topology is not Hausdorff.

By definition the map π is continuous. Since $\mathbb{R}$ is connected, and π surjective, it follows that Y is connected.
Alternative approach: One may use the description under (b) as follows. Let $U, V \in \mathscr{T}_Y$ and assume $Y = U \cup V$, $U \cap V = \emptyset$. Without loss of generality we may assume that $b \in U$. Then $U = Y$ which forces $V = 0$. Hence, Y is connected for the quotient topology.

Since Y is finite, every open cover of Y is already finite, hence Y is compact.

Solution 3.

(a) Assume (1). Then without loss of generality we may assume that X_1 is compact. Since X^+ is Hausdorff, X_1 is closed in X^+. Thus, $X^+ \setminus X_1$ is open in X^+ and contains X_2 hence is non-empty. Also, X_1 is open in X^+ and non-empty. We find that X^+ is the disjoint union of two open non-empty subsets X_1 and $X^+ \setminus X_1$, hence not-connected.

(b) It follows from the assumption that $U \cap X_j$ is both open and closed in X_j. As U is the union of these intersections, one of them is non-empty. Without loss of generality we may assume that $U \cap X_1 \neq \emptyset$. Now X_1 is the disjoint union of the open subsets $U \cap X_1$ and $X_1 \setminus (U \cap X_1)$. By connectedness of X_1, the second set must be empty, hence $U \cap X_1 = X_1$, so that $X_1 \subset U$.

(c) Assume (2). Then there exist non-empty open sets $U, V \subset X^+$ which are disjoint and such that $U \cup V = X^+$. As U, V are each other's complement, they are closed in X^+ as well. Hence they are also compact.

Without loss of generality we may assume that $\infty \in V$ so that $U = X^+ \setminus V$ is a subset of X. Since the topology on X is induced by the topology on X^+, it follows that U is open, closed and compact in X. By item (b) we may assume that X_1 is contained in U. Since U is compact and X_1 closed in U it follows that X_1 is compact.

(d) Let $X := (-2,-1) \cup (0,1)$, equipped with the restriction topology of the Euclidean topology on $\mathbb{R}$. Since X is the disjoint union of two non-empty open subsets, it is not connected. Thus $X_1 = (-2,-1)$ and $X_2 = (0,1)$ are as in the above, and non-compact. It follows that X^+ is connected.

Solution 4.

(a) Since X is a subspace of a Hausdorff space, it is Hausdorff. As X is the union of the two closed and bounded subsets $D \times \{-1\}$ and $D \times \{+1\}$, the set X is closed and bounded in $\mathbb{R}^3$, hence compact.

(b) We note that $\|\varphi(x, \pm 1)\|^2 = \|x\|^2 + (1 - \|x\|^2) = 1$, hence φ maps into the unit sphere. If y is a point of the unit sphere, we may write $y = (x,t)$, with $x \in \mathbb{R}^2$ and $t \in \mathbb{R}$ and then $\|x\|^2 + t^2 = \|y\|^2 = 1$ so that $\|x\|^2 \leq 1$ and $t^2 = (1 - \|x\|^2)$. It follows that $x \in D$ and $t = \pm\sqrt{1 - \|x\|^2}$. Hence $y = \varphi(x, \pm 1)$. This shows that φ is surjective.

(c) If f and g belong to A, then $(f+g)(x,-1) = f(x,-1) + g(x,-1) = f(x,1) + g(x,1) = (fg)(x,1)$ for all $x \in \partial D$. Hence $f + g \in A$. Similarly one shows that $fg \in A$. If $\lambda \in \mathbb{R}$ and $f \in A$ then for $x \in \partial D$ we have $\lambda f(x,-1) = \lambda f(x,-1) = \lambda f(x,1) = (\lambda f)(x,1)$ and we see that $\lambda f \in A$. Finally, the constant function 1 belongs to A. It follows that A is a unital subalgebra.

(d) We will determine the fibers $\varphi^{-1}(y)$ of the map φ. First, let $y = (x,t)$ be a point of the unit sphere with $t \neq 0$. Then it follows from the reasoning in (b) that $(x, \text{sign}(t).1)$ is the unique element in the fiber $\varphi^{-1}(y)$. Next, let $y = (x,t)$ be in the unit sphere and assume that $t = 0$. Then it follows that $\|x\| = 1$ and $t = 0$, and we see that $\varphi(x', \eta) = (x, 0)$ if and only if $x' = x$ and $\eta \in \{-1, 1\}$, hence $\varphi^{-1}(y)$ consists of the points $(x, \pm 1)$.

It follows from the above that A is precisely the algebra of continuous functions $f : X \to \mathbb{R}$ which are constant on the fibers of φ. It follows that $\varphi^* : f \mapsto f \circ \varphi$ is a bijection from $C(S^2)$ onto A. This bijection is an isomorphism of algebras. Thus, the algebras A and $C(S^2)$ are isomorphic and from this we infer that the topological spectrum $\mathbf{X}_A$ is homeomorphic to the topological spectrum of $C(S^2)$. By the Gelfand-Naimark theorem, the latter is homeomorphic to S^2. Thus, $\mathbf{X}_A$ is homeomorphic to S^2.