Retake Inleiding Topologie, 18/4-2017, 13:30 - 16:30

Solution 1.

(a) The sets $\emptyset$ and $\mathbb{R}$ belong to $\mathscr{T}$. If $U,V \in \mathscr{T}$ then $\mathbb{R}\setminus(U\cap V) = (\mathbb{R}\setminus U)\cup(\mathbb{R}\setminus V)$ is either $\mathbb{R}$ or finite, hence $U\cap V$ belongs to $\mathscr{T}$. If $\{U_i \mid i\in I\}$ is a collection of sets in $\mathscr{T}$, then the union $U = \cup_i U_i$ has complement $\mathbb{R}\setminus U = \cap_{i\in I}(\mathbb{R}\setminus U_i)$. If all sets U_i are empty, then so is U hence U is open in that case. In the remaining case, at least one of $\mathbb{R}\setminus U_i$ is finite, hence $\mathbb{R}\setminus U$ is finite and we conclude that $U\in\mathscr{T}$.

(b) Let U_0 and U_1 be any open sets with $U_0 \ni 0$ and $U_1 \ni 1$. Then it follows that the set $\mathbb{R}\setminus(U_0\cap U_1) = (\mathbb{R}\setminus U_1)\cup(\mathbb{R}\setminus U_2)$ is finite, hence its complement $U_1\cap U_2$ is non-empty. Hence 0 and 1 cannot be separated and we see that topology is not Hausdorff.

(c) The closed sets of $\mathbb{R}$ with respect to $\mathscr{T}$ are precisely the finite sets, and $\mathbb{R}$. Thus, the only closed set containing $\mathbb{Z}$ is $\mathbb{R}$, and it follows that the closure of $\mathbb{Z}$ is $\mathbb{R}$.

(d) Let $U\subset[0,1]$ be a set of $\mathscr{T}$. Then the complement of U is infinite, hence $U=\emptyset$. We conclude that the only set of $\mathscr{T}$ contained in $[0,1]$ is the empty set. Hence the interior of $[0,1]$ is empty.

(e) Let S be a subset of $\mathbb{R}$ and let $\{S_i \mid i\in I\}$ be an open cover of S for the induced topology. Then every S_i is of the form $S\cap U_i$, where $S_i\in\mathscr{T}$. If S is the emptyset, there is nothing to prove. Thus, assume S contains a point x. Then $x\in U_{i_0}$ for some i_0. It follows that $\mathbb{R}\setminus U_{i_0}$ is finite, so $S\setminus S_{i_0} = S\setminus U_{i_0}$ is finite hence consists of elements $x_1,\ldots,x_N$. For each $1\le k\le N$ chose $i_k\in I$ such that $x_k\in S_k$. Then the sets $S_{i_0},\ldots,S_{i_N}$ cover S. It follows that S is compact.

(f) Assume that A is not connected. Then $A = A_1\cup A_1$, with A_1 and A_2 disjoint non-empty open subsets of A for the induced topology. Then there exist open U_j of X such that $A_j = A\cap U_j$. Clearly, U_j is non-empty, hence $\mathbb{R}\setminus U_j$ is finite. It follows that $A_1 = A\setminus A_2 = A\setminus U_2 \subset \mathbb{R}\setminus U_2$ hence A_1 is finite. Likewise, A_2 is finite. It follows that $A = A_1\cup A_2$ is finite.

Conversely, assume that A is finite. Select $a\in A$ and write $A_1 = \{a\}$ and $A_2 = A\setminus A_1$. Then A_2 is finite in $\mathbb{R}$ hence closed. Hence A_1 and A_2 are two closed subsets of A whose disjoint union is A. It follows that A_1 and A_2 are open in A as well, hence A is not connected.

Solution 2.

(a) Let $x\in X$. Then there exists an open neighborhood $U_x\ni x$ such that $f|_{U_x}$ is injective. It follows that for every $y\in Y$ there can be at most one $x'\in U_x$ such that $f(x')=y$. Hence, $f^{-1}(\{y\})\cap U_x$ has at most one element. Thus, $\{U_x \mid x\in X\}$ is an open covering of X as asserted.

(b) Let $\{U_i \mid i \in I\}$ be a covering as mentioned in (a). Then there exist finitely many indices $i_1, \ldots, i_N$ such that $U_{i_1}, \ldots, U_{i_N}$ cover X. Let $y \in Y$. Then

$$f^{-1}(y) = f^{-1}(y) \cap (U_{i_1} \cup \cdots \cup U_{i_N}) \subset \bigcup_{k=1}^{N} (f^{-1}(y) \cap U_{i_k}).$$

In view of (a), this implies that $\# f^{-1}(\{y\}) \leq N$.

Solution 3.

(a) Let $i \in I$. Then $A \cap U_i$ is open and closed in U_i for the induced topology. Its complement in U_i is $U_i \setminus A$, and is closed and open in U_i. It follows that U_i is the disjoint union of the open subsets $U_i \cap A$ and $U_i \setminus A$. One of these sets must be empty since U_i is connected. If the second set is empty, then $U_i \subset A$ hence $U_i \cap A = U_i$. The assertion follows.

(b) Assume that A is not disjoint from U_i. Then it follows from (a) that A contains U_i. It follows that A contains $U_i \cap U_j$ hence is not disjoint from U_j. Again by (a) it follows that A contains U_j. Hence A contains both U_i and U_j.

Likewise, if $A \cap U_j \neq \emptyset$ then A contains both U_j and U_i. The result follows.

(c) Let $i \sim j$. There exists a sequence $i_0, \ldots, i_n$ in I such that $i_0 = i$, $i_n = j$ and $U_{i_{k-1}} \cap U_{i_k} \neq \emptyset$ for all $1 \leq k \leq n$. Assume $A \cap U_i \neq \emptyset$. Then it follows by applying (b) repeatedly that $U_{i_k} \subset A$ for all $0 \leq k \leq n$. In particular, both U_i and U_j are contained in A. Likewise, if $U_j \cap A \neq \emptyset$, then A contains both U_j and U_i.

(d) Assume that X is not connected. Then X can be written as the disjoint union of two non-empty open sets A_1 and A_2. Then A_1 is open and closed in X and non-empty. Since the U_i cover X it follows that there exists $i \in I$ such that $U_i \cap A_1 \neq \emptyset$. Since A_1 is a proper subset of X there must be j such that $A_1 \not\supset U_j$. By (c) it follows that j is not equivalent to i. The assertion now follows by contraposition.

(e) Arguing by contraposition, assume that not all elements of I are equivalent. Let $i_1 \in I$ be such that $U_i \neq \emptyset$ and let I_1 be the equivalence class of i_1. Let A_1 be the union of the sets U_i for $i \in I_1$. Then A_1 is open. If $j \not\sim i_1$ then it follows that $U_j \cap U_i = \emptyset$ for all $i \in I_1$ hence $U_j \cap A_1 = \emptyset$. Thus the union A_2 of sets U_j for $j \in I \setminus I_1$ is non-empty, open and disjoint from A_1. Obviously $A_1 \cup A_2 = X$. It follows that X is not connected.

Solution 4.

(a) Let $w, z \in \bar{D}$ be distinct and assume that zRw. Then $\varphi(z) = \varphi(w)$. Hence, $z^2 = w^2$, and we find $-z = w$, in particular $|z| = |w|$ and $z \neq -z$. By looking at the first components of $\varphi(z)$ and $\varphi(w)$ we see that $(1-|z|)z = (1-|w|)w = -(1-|z|)z$ hence $(1-|z|)2z = 0$ and we see that $|z| = 1$. Thus, if $z, w \in \bar{D}$ are distinct then zRw implies $|z| = 1$ and $w = -z$.

Conversely, assume that $|z| = 1$ and $z = -w$. Then it readily follows that $z \neq w$ and $\varphi(z) = \varphi(w)$. Thus, we see that for different $z, w \in \bar{D}$ we have zRw if and only if $z \in \partial\bar{D}$ and $w = -z$.

It follows from this that $\bar{D}/R$ equipped with the quotient topology is homeomorphic to $\mathbb{P}^2(\mathbb{R})$.

(b) The map $\varphi : \bar{D} \to \mathbb{C}^2$ is continuous, hence factors through an injective continuous map $\bar{\varphi} : \bar{D}/R \to \mathbb{C}^2$. Since $\bar{D}$ is compact, so is its continuous image $\bar{D}/R$ and since $\mathbb{C}^2$ is Hausdorff the map $\bar{\varphi}$ is a topological embedding. Since $\bar{D}/R$ is homeomorphic to $\mathbb{P}^2(\mathbb{R})$ and $\mathbb{C}^2$ is homeomorphic to $\mathbb{R}^4$, it follows that there exists a topological embedding of $\mathbb{P}^2(\mathbb{R})$ into $\mathbb{R}^4$.

(c) Let $p : \bar{D} \to \bar{D}/R$ be the natural projection. Then the map $p^* : C(\bar{D}/R) \to C(\bar{D}), f \mapsto f \circ p$ is an injective homomorphism of algebras with image A. It follows that the algebra A is isomorphic with the algebra $C(\bar{D}/R)$. It follows that the topological spectrum $\mathbf{X}_A$ is homeomorphic to the topological spectrum of $C(\bar{D}/R)$ which in turn is homeomorphic to $\bar{D}/R \simeq \mathbb{P}^2(\mathbb{R})$.

Solution 5.

(a) Assume that $\widehat{f} : \widehat{X} \to \widehat{Y}$ is continuous. Let $K \subset Y$ be compact. Then $V := \widehat{Y} \setminus K$ is open in $\widehat{Y}$ hence its preimage $U := f^{-1}(V)$ is open in $\widehat{X}$. Since V contains ∞_Y, the open set U contains $f^{-1}(\infty_Y) = \infty_X$, hence its complement $\widehat{X} \setminus U$ is closed hence compact, and contained in X. We now note that $f^{-1}(K) = \widehat{f}^{-1}(K) = \widehat{f}^{-1}(\widehat{Y} \setminus V) = \widehat{X} \setminus U$ is compact in X.

(b) Assume that for every compact $K \subset Y$ the preimage $f^{-1}(K)$ in X is compact for the relative topology. Let $V \subset \widehat{Y}$ be an open subset.

Case 1: V does not contain ∞_Y. Then V is contained in Y hence $\widehat{f}^{-1}(V)$ equals $f^{-1}(V)$ hence is open in X by continuity of f. Since X is open in $\widehat{X}$ it follows that $f^{-1}(V)$ is open in $\widehat{X}$.

Case 2: $V \ni \infty_Y$. In this case $K := \widehat{Y} \setminus V$ is closed in $\widehat{Y}$ hence compact. Furthermore, K is contained in Y hence $f^{-1}(K)$ is a compact subset of X. It follows that $\widehat{f}^{-1}(K) = f^{-1}(K)$ is compact in X hence in $\widehat{X}$. Since the latter is Hausdorff, $\widehat{f}^{-1}(K)$ is closed in $\widehat{X}$ and we find that $\widehat{f}^{-1}(V) = \widehat{f}^{-1}(\widehat{Y} \setminus K) = \widehat{X} \setminus \widehat{f}^{-1}(K)$ is open in $\widehat{X}$.

It follows that in all cases $\widehat{f}^{-1}(V)$ is open in $\widehat{X}$. Hence, $\widehat{f}$ is continuous and the converse implication has been established.