Tentamen Inleiding Topologie, WISB243 2019-01-29, 13:30 – 16:30

Solution 1

(a) Since $\mathbb{R} \notin \mathscr{B}$, the collection $\mathscr{B}$ is not a topology. To see that it is a topology basis, let B_1, B_2 be two elements of B and assume that $x \in B_1 \cap B_2$. Then $B_j = (a_j, n_j]$ with $n_j \in \mathbb{Z}$ and $a_j \in \mathbb{R}$. We have the inequalities $a_j < x \leq n_j$ Hence, $a := \max(a_1, a_2) < x \leq \min(n_1, n_2) =: n$. Clearly, $x \in (a, n] \subset B_1 \cap B_2$. It follows that B is a basis.

(b) Let $\mathscr{B}_0 = \{(q, n] \mid n \in \mathbb{Z}, q \in \mathbb{Q}, q < n\}$. Then $\mathscr{B}_0$ is a countable subset of $\mathscr{B}$. If $B \in \mathscr{B}$, then $B = (a, n]$ for $a \in \mathbb{R}$ and $n \in \mathbb{Z}$. Clearly B is the union of the sets $(q, n] \in \mathscr{B}_0$ with $q \in \mathbb{Q}$, $a < q < n$. It follows that $\mathscr{B}_0$ is a basis for $\mathscr{T}$. As $\mathscr{B}_0$ is countable, it follows that $\mathscr{T}$ is second countable.

Clearly, if $B := (a, n] \in \mathscr{B}$ contains $\frac{1}{2}$, then $n \geq 1$ and it follows that $1 \in B$. Hence 1 and $\frac{1}{2}$ cannot be separated by sets from the basis, and therefore not by open sets. We thus see that $(\mathbb{R}, \mathscr{T})$ is not Hausdorff. Since every metric space is Hausdorff, $(\mathbb{R}, \mathscr{T})$ cannot be metrizable.

(c) If $x \in (1, \infty)$, then $x \in (1, n]$ for some $n \in \mathbb{Z}$. As the latter set is disjoint from A, it follows that x does not belong to the closure of A.

If $x \in (0, \frac{1}{2})$ then each set from $\mathscr{B}$ containing x is of the form $B = (a, n]$ with $n \geq 1$ and $a < \frac{1}{2}$. Each such set B intersects A, so that x is in the closure of A. On the other hand, if $x < 0$, then $x \in B := (x - 1, 0] \in \mathscr{B}$, and $B \cap A = \emptyset$. It follows that such x is not in the closure of A. We conclude that the closure of A equals $(0, 1]$.

(d) Let $\mathscr{U}$ be an open cover of A by sets from $\mathscr{T}$. Then there is a set $U \in \mathscr{U}$ which contains $\frac{1}{2}$. Since $\mathscr{B}$ is a basis, there is a $B \in \mathscr{B}$ such that $\frac{1}{2} \in B \subset U$. There exist $n \in \mathbb{Z}$ and $a \in \mathbb{R}$ such that $B = (a, n]$, hence $n \geq 1$ and $a < \frac{1}{2}$. It follows that $A \subset B \subset U$. We conclude that $\mathscr{U}$ contains a finite subcover (consisting of just U). Hence, A is compact.

We define $U_0 = (-1, 0]$ and for $j \geq 1$ we define $U_j = (\frac{1}{j}, 1]$. Then $\{U_j\}_{j \geq 0}$ is a cover of $[0, 1]$. Any finite subcover is contained in a union $U_0 \cup U_1 \cup \ldots \cup U_N$ which does not contain the point $1/N$ of $[0, 1]$. Hence the given cover has no finite subcover, and we conclude that $[0, 1]$ is not compact.

Solution 2

(a) It is easy to see that $\mathbb{R}^2 \setminus \{(0, 0)\}$ is arcwise connected. Hence it is connected.

(b) Suppose there exists a continuous injective map $f : \mathbb{R}^2 \to \mathbb{R}$. Then by continuity and connectedness of f it follows that $f(\mathbb{R}^2)$ is connected, hence an interval I in

$\mathbb{R}$. This interval has at least two points, so contains an interval of the form (a,b) with $a<b$. Take $c \in (a,b)$, then $I \setminus \{c\}$ is not an interval, hence not connected. On the other hand, $c = f(p)$ for a point $p \in \mathbb{R}^2$. By injectivity, it follows that $f(\mathbb{R}^2 \setminus \{p\}) = I \setminus \{c\}$. Since f is continuous and $\mathbb{R}^2 \setminus \{p\}$ arcwise connected, it follows that $f(\mathbb{R}^2 \setminus \{p\})$ is connected, contradiction.

(c) Let $g : M \to S^1$ be a continuous injective map. Since M is a topological manifold, there exists a continuous map $\varphi : \mathbb{R}^2 \to M$ which is a homeomorphism onto an open subset $U \subset M$. Replacing $\mathbb{R}^2$ by an open ball, which is homeomorphic to $\mathbb{R}^2$, we see that we may arrange that φ is not surjective. Select $m \in M - \varphi(M)$, then by injectivity of g it follows that $g \circ \varphi : \mathbb{R}^2 \to S^1 - \{g(m)\}$ is injective continuous. There exists a homeomorphism $\psi : S^1 - g(m) \to \mathbb{R}$. We now see that $\psi \circ g \circ \varphi : \mathbb{R}^2 \to \mathbb{R}$ is injective continuous, contradicting (a).

Solution 3

(a) Let $x \in X$. Then $\pi(x) \in \pi(A)$ if and only if $\pi(x) = \pi(a)$ for some $a \in A$. This implies that $x \in A$. We thus see that $\pi^{-1}(\pi(A)) \subset A$. The converse inclusion is obvious.

(b) Let $x \in X$. Then $\pi(x) \in \pi(S)$ implies that $\pi(x) = \pi(y)$ for an $y \in S$. Since $y \notin A$ it follows that $x = y \in S$. Hence $\pi^{-1}(\pi(S)) \subset S$. Again, the converse inclusion is obvious.

(c) Write $T = A \cup S$ with $S = T \setminus A$. Then $\pi(T) = \pi(A) \cup \pi(S)$ so $\pi^{-1}(\pi(T)) = \pi^{-1}(\pi(A)) \cup \pi^{-1}(\pi(S)) = A \cup S = T$.

(d) Let $X = U \cup V$ be a partition by open subsets. Then $A = (A \cap U) \cup (A \cap V)$ is a disjoint union of open sets for the induced topology. Since A is connected, one of the sets in the union must be empty. Without loss of generality, assume that $A \cap V = \emptyset$. Then $A \subset U$ and we see that $\pi^{-1}(\pi(U)) = U$ and $\pi^{-1}(\pi(V)) = V$. By definition of the quotient topology, it follows that both $\pi(U)$ and $\pi(V)$ are open subsets of X/A. Moreover, they are disjoint. For if $\pi(U) \cap \pi(V)$ were not empty, then by surjectivity of π,

$$\emptyset \subsetneq \pi^{-1}(\pi(U) \cap \pi(V)) = U \cap V,$$

contradiction.

(e) If X is connected, the so is its image X/A under the continuous map π.

For the remaining implication, suppose that X is not connected. Then there exists a partition $X = U \cup V$ by open subsets. It follows that $X/A = \pi(U) \cup \pi(V)$, which union of non-empty open sets. The union is disjoint by (d). We conclude that X/A is not connected.

Solution 4

(a) Each V_i is open in A for the induced topology. By definition this means that $V_i = A \cap U_i$ for a suitable open subset U_i of X.

(b) Let $x \in X$. If $x \in A$ then there exists $i \in I$ such that $x \in V_i$, hence $x \in U_i$. On the other hand, if $x \notin A$, then $x \in X - A$. It follows that $\mathscr{U}^*$ is an open covering of X.

(c) The locally finite refinement exists by the definition of paracompactness of X.

(d) Since $\mathscr{W}$ is a cover of X it follows that

$$A = \cup_{j \in J} (A \cap W_j).$$

The sets $A \cap W_j$ are empty for $j \in J_0$. Hence $\mathscr{W}'$ is an open cover of A.

Finally, let $j \in J_1$, then W_j is contained in a set $U \in \mathscr{U}^*$. Since $j \notin J_0$ it follows that $U \neq X - A$. Hence, there exists $i \in I$ such that $W_j \subset U_i$. It follows that

$$W_j \cap A \subset U_i \cap A = V_i.$$

Hence $\mathscr{W}'$ is subordinate to $\mathscr{V}$.

(e) We will show that the cover $\mathscr{W}'$ in (d) is locally finite. If $x \in A$ then there exists a neighborhood N of x in X such that the set $J_N := \{j \in J \mid N \cap W_j \neq \emptyset\}$ is finite. Now $N \cap A$ is a neighborhood of x in A. Furthermore, $N \cap A \cap W_j \neq \emptyset$ implies $j \notin J_0$ and $N \cap W_j \neq \emptyset$ hence $j \in J_N \cap J_1$. It follows that $\mathscr{W}'$ is locally finite.

We have thus shown that every open cover of A admits a locally finite refinement. Therefore, A is paracompact.