Retake Inleiding Topologie, WISB243 2018-04-16, 13:30 – 16:30

Solution 1

(a) If $U \in \mathscr{T}$ and $c \in \mathbb{R}$ the pre-image $T_c^{-1}(U)$ belongs to $\mathscr{T}$ by continuity of T_c. The pre-image is the set of $x \in \mathbb{R}$ such that $x + c \in U$, which equals $U + (-c)$. Thus, if we take $c = -a$ we see that $U + a \in \mathscr{T}$.

(b) If $U \in \mathscr{T}$ then also the set $\frac{1}{2}U = V^{-1}(U)$ belongs to $\mathscr{T}$. By applying this repeatedly to the set $[0,1)$ we see that $[0, 2^{-n}) \in \mathscr{T}$ for every $n \in \mathbb{N}$. By applying (a) we now find that $[a, a+2^{-n}) \in \mathscr{T}$ for all $a \in \mathbb{R}$ and $n \in \mathbb{N}$.

(c) Each set $[a, a+1)$ belongs to $\mathscr{B}$, so $\cup\mathscr{B} = \mathbb{R}$. Assume $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$. We will show that there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. It is easily seen that $B_1 \cap B_2$ is either empty or a set of the form $[a, b)$, with $a < b$. Let $x \in [a, b)$. Then there exists $n \in \mathbb{N}$ such that $x + 2^{-n} < b$. Put $B_3 := [x, x+2^{-n})$, then it is clear that B_3 satisfies all assertions.

(d) Let $\mathscr{T}_0$ be the topology generated by $\mathscr{B}$. Then $\mathscr{T}_0 \subset \mathscr{T}$. On the other hand, if $B \in \mathscr{B}$ then it is readily verified that $V^{-1}(B) \in \mathscr{B}$ and $T_c^{-1}(B) = B - c \in \mathscr{B}$ for all $c \in \mathbb{R}$. Hence, V and T_c are continuous for $\mathscr{T}_0$. Also, $[0,1) \in \mathscr{T}_0$, so $\mathscr{T}_0$ satisfies the properties (1), (2), (3). Since $\mathscr{T}$ is the smallest topology with this property, it follows that $\mathscr{T}_0 = \mathscr{T}$.

Solution 2

(a) Let $x = (z,t) \in S \times J$. The orbit Γx consists of the points (z,t) and $(-z,-t)$. Since $\|z\| = 1$, $z \neq -z$, so (z,t) and $(-z,-t)$ are distinct points.

(b) Let $\xi \in X/\Gamma$ and let (z,τ) be in the fiber of ξ. By replacing (z,τ) with $(-z,-\tau)$ if necessary, we see that (z,τ) can be found with $z_2 \geq 0$. It follows that there exists $\varphi \in [0,\pi]$ such that $z = (\cos\varphi, \sin\varphi)$. Take $s = \varphi/\pi$ and $t = (\tau+1)/2$, then $0 \leq s,t \leq 1$ and $\sigma(s,t) = p(z,\tau) = \xi$. Hence σ is surjective.

(c) Since σ is constant on the classes of $\sim$ there is a unique map $\bar{\sigma} : [0,1]^2/\sim \to X/\Gamma$ such that $\bar{\sigma} \circ q = \sigma$. Clearly, $\bar{\sigma}$ is bijective and continuous. Since $[0,1]^2$ is compact, and q continuous, $[0,1]^2/\sim$ is compact. Since X is Hausdorff, and Γ finite, X/Γ is Hausdorff. It follows that $\bar{\sigma}$ is an embedding. As $\bar{\sigma}$ is surjective, it is a homeomorphism.

(d) Assertion (1) is incorrect, assertion (2) is correct.

Solution 4

(a) First assume that f is locally constant and let $z \in Z$. If $y \in f^{-1}(\{z\})$ then there exists an open neighborhood U of y such that f is constant on U hence $f = f(y) = z$ on U, so that $U \subset f^{-1}(\{z\})$. It follows that every point of the fiber $f^{-1}(\{z\})$ is interior, hence the fiber is open.

Conversely, assume every fiber is open, and let $y \in Y$. Put $z = f(y)$ and $U := f^{-1}(\{z\})$. Then U is open and contains y. Furthermore, f is constant on U. We see that f is locally constant.

(b) First assume that f is locally constant. Let $V \subset Z$ be any subset. Then $f^{-1}(V)$ is the union of the fibers $f^{-1}(\{z\})$, for $z \in V$. All fibers are open by (a), hence $f^{-1}(V)$ is open. It follows that f is continuous.

Conversely, assume that f is continuous for the discrete topology on Z. Then for every $z \in Z$ the set $\{z\}$ is open for the discrete topology, hence $f^{-1}(\{z\})$ is open. It follows from (a) that f is locally constant.

(c) By (b), we know that f is continuous for the discrete topology on Z. It follows that $f(Y)$ is connected for the induced topology on $f(Y)$, which is the discrete topology. If $z \in f(Y)$ then $\{z\}$ and $f(Y) \setminus \{z\}$ are disjoint open subsets of $f(Y)$, and we see that one of them must be empty. Hence $f(Y) = \{z\}$ and we see that f is constant.

Alternative reasoning: Let Y be connected, and $f : Y \to Z$ locally constant. If $Y = \emptyset$, there is nothing to prove. Select $y \in Y$ and put $z := f(y)$. Then the fiber $U_1 := f^{-1}(z)$ is open by (a) and contains y hence is not empty. Its complement U_2 in Y equals $f^{-1}(Z \setminus \{z\})$ which is open by (b). From the connectedness of Y we conclude that $U_2 = \emptyset$. Hence $U_1 = Y$ and we see that $f(Y) = \{z\}$, hence, f is constant.

(d) Assume that Y is not connected. Then $Y = U \cup V$ for certain disjoint non-empty open subsets U and V of Y. We take $Z = \{0, 1\}$ and define $f(y) = 0$ if $y \in U$ and $f(y) = 1$ for $y \in V$. Then clearly, f is locally constant, but not constant.

Solution 3

(a) For every $a \in X$ there exists an open neighborhood U_a and a constant $m_a > 0$ such that $f_a \geq m_a$ on U_a.

The U_a, for $a \in X$, cover X. By compactness there exists a finite collection $a_1, \ldots, a_n$ of points of X such that $X \subset \cup_{j=1}^n U_{a_j}$. It follows that $f \geq \min_{1 \leq j \leq n} m_{a_j}$ on each of the sets U_i hence on X. Hence, (a) is valid, with $m_X := \min_{1 \leq j \leq n} m_{a_j}$.

(b) For every $x \in X$ there exists an open neighborhood U_x of x in X and a constant $m_x > 0$ such that $f \geq m_x$ on U_x. The open sets U_x, for $x \in X$, form an open covering of X. By paracompactness, this cover has a locally finite refinement, $\{V_i \mid i \in I\}$. By definition of refinement, for every $i \in I$ there exists an $x_i \in X$ such that $V_i \leq U_{x_i}$. It follows that $f \geq m_i := m_{x_i}$ on V_i.

(c) Let V_i, m_i be as above. Then there exists a partition of unity $\{\eta_i \mid i \in I\}$ which is subordinated to $\{V_i \mid i \in I\}$. Now $0 \leq \eta_i \leq 1$ and $\eta_i = 0$ outside V_i. Therefore, $\eta_i f \geq m_i \eta_i$ on V_i and on $X \setminus V_i$ hence on X. We now note that for every $x \in X$ we have (with finitely many nonzero terms)

$$f(x) = \sum_{i \in I} \eta_i(x) f(x) \geq \sum_{i \in I} m_i \eta_i(x).$$

As the family $\{\eta_i \mid i \in I\}$ is locally finite, the sum

$$\mu := \sum_i m_i \eta_i$$

is a locally finite sum of continuous functions, hence continuous. Since $\sum_i \eta_i = 1$, we have $\mu > 0$ everywhere. Hence μ is a continuous function $X \to (0, \infty)$ and we have shown that $f \geq \mu$ on X.