Exam Inleiding Topologie, WISB243 **2018-01-29, 13:30 – 16:30**

Solution to 1

(a) Let $B_1, B_2 \in \mathscr{B}$. If one of B_1, B_2 equals $\mathbb{R}$, then obviously $B_1 \cap B_2 \in \mathscr{B}$. Assume that B_1, B_2 are not equal to $\mathbb{R}$. Then $B_j = [n_j, a_j)$, with $n_1, n_2 \in \mathbb{Z}$ and $a_1, a_2 \in \mathbb{R}$. It is now readily seen that $B_1 \cap B_2 = [n, b)$ with $n = \max(m_1, m_2)$ and $b = \min(a_1, a_2)$. Hence $B_1 \cap B_2 \in \mathscr{B}$. This shows that $\mathscr{B}$ is a topology basis. Since $\cup_{m<0}[m, 0) = (-\infty, 0) \notin \mathscr{B}$, we see that $\mathscr{B}$ is not closed under unions. It follows that $\mathscr{B}$ is not a topology.

(b) Let $U \in \mathscr{T}$ contain $\frac{1}{2}$. Then there exists $[m, a) \in \mathscr{B}$ with $\frac{1}{2} \in [m, a) \subset U$. We must have $m \leq 0$ and $a > \frac{1}{2}$, hence $[0, a) \subset U$. In particular, $0 \in U$. It follows that 0 and $\frac{1}{2}$ cannot be separated by open neighborhoods. Hence, $\mathscr{T}$ is not Hausdorff.

(c) The subset $\mathscr{B}_0 \subset \mathscr{B}$ consisting of all intervals $[m, q)$ with $m \in Z$ and $q \in \mathbb{Q}$ is countable. Moreover, if $a > 0$ then $[m, a) = \cup_{q \in \mathbb{Q}, q < a}[m, q)$, so $\mathscr{B}_0$ is a countable basis for $\mathscr{T}$. It follows that $\mathscr{T}$ is second countable.

(d) A non-empty basis element $[m, a) \in \mathscr{B}$ is contained in A if and only if $m \geq 0$ and $a \leq \frac{1}{2}$. The latter is equivalent to $m = 0$ and $a \leq \frac{1}{2}$. The union of these sets is $\mathrm{Int}(A) = [0, \frac{1}{2})$.

The condition $x \notin \overline{A}$ is equivalent to the existence of $m \in \mathbb{Z}$ and $a \in \mathbb{R}$ with $x \in [m, a)$ and $[m, a) \cap A = \emptyset$. The latter condition forces $m \geq 1$ or $a < -\frac{1}{2}$ and we see that $x \notin \overline{A}$ implies $x \in [1, \infty)$ or $x \in (-\infty, -\frac{1}{2})$. Conversely, if $x \in [1, \infty)$ or $x \in (-\infty, -\frac{1}{2})$ then either $x \in [1, a)$ for $a > 1$ or $x \in [m, -\frac{1}{2})$ for $m \leq -1$. In both cases, there exist $m \in Z$ and $a \in \mathbb{R}$ such that $x \in [m, a)$ and $[m, a) \cap A = \emptyset$. We conclude that $\overline{A}$ equals the complement of $[1, \infty) \cup (-\infty, -\frac{1}{2})$ which equals $[-\frac{1}{2}, 1)$.

(e) Assume that $0 < r < 1$. Any open subset U of $[0, r]$ containing r must contain a subset of the form $[0, r] \cap [m, a)$, for $m \leq r < a$. The latter implies $m \leq 0$ and $a > r$ hence $[0, r] \subset [0, r] \cap [0, a) \subset U$ hence $U = [0, r]$. This implies that $[0, r]$ cannot be written as the union of two disjoint non-empty open subsets. Hence $[0, r]$ is connected.

Now assume that $r \geq 1$. Then $[1, r] = [0, r] \cap [1, r+1)$ hence $[1, r]$ is open and non-empty in $[0, r]$. Obviously, $[0, 1)$ is open and non-empty in $[0, r]$ and $[0, r]$ is the disjoint union of $[0, 1)$ and $[1, r]$. It follows that $[0, r]$ is not connected.

Solution to 2

(a) We assume that both X and Y are Hausdorff. Let $a, b \in X \times Y$ be two points such that $a \neq b$. Write $a = (a_1, a_2)$ and $b = (b_1, b_2)$, then we may as well assume that $a_1 \neq b_1$. By the Hausdorff property of X there exist open subsets $U, V \subset X$ such that $a_1 \in U$, $a_2 \in V$ and $U \cap V = \emptyset$. Now $U \times Y$ and $V \times Y$ are open subsets of $X \times Y$ containing a and b respectively, and

$$U \times Y \cap V \times Y = (U \cap V) \times Y = \emptyset.$$

It follows that the product is Hausdorff.

(b) For the converse, assume that $X \times Y$ is Hausdorff. Let $a_1, b_1 \in X$ be distinct points. Select a point $y \in Y$ then (a_1, y) and (b_1, y) are distinct points in $X \times Y$. By the Hausdorff property, there exist open subsets W_1, W_2 in $X \times Y$ such that $(a_1, y) \in W_1$, $(b_1, y) \in W_2$ and $W_1 \cap W_2 = \emptyset$. Since W_1 is open, there exists an open subset $U_1 \ni a_1$ of X such that $U_1 \times \{y\} \subset W_1$. Likewise, there exists an open subset $U_2 \ni b_1$ of X such that $U_2 \times \{y\} \subset W_2$. We now observe that

$$(U_1 \cap U_2) \times \{y\} = U_1 \times \{y\} \cap U_2 \times \{y\} \subset W_1 \cap W_2 = \emptyset.$$

It follows that $U_1 \cap U_2 = \emptyset$. We conclude that a_1, b_1 are separated in X. Hence, X is Hausdorff. In a similar way, it follows that Y is Hausdorff.

Solution to 3

1. We first show that '(2) $\Rightarrow$ (1)'. Let $g : [0, \infty) \to \mathbb{R}$ be a continuous function such that $f \leq g$. Let $a \in X$. Let $U = g^{-1}((-\infty, g(a) + 1))$. Then by continuity of g it follows that U is open. Clearly $a \in U$. Furthermore, $g \leq g(a) + 1$ on U. It follows that $f \leq M$ on U, with $M = g(a) + 1$.

2. We now address the converse implication '(1) $\Rightarrow$ (2)'. Assume that f is locally bounded. Since X is locally compact Hausdorff and second countable, it is paracompact.

   For every $a \in X$ there exists an open neighborhood V_a of a such that f is bounded on V_a by a suitable constant $M_a > 0$. Let $\mathscr{V}$ be a collection of such open neighborhoods V_a, for $a \in X$.

   First reasoning. Then by paracompactness, $\mathscr{V}$ has a locally finite refinement $\mathscr{U} = \{U_i \mid i \in I\}$. For every $i \in I$ the neighborhood U_i is contained in a neighborhood $V_{a(i)}$ for a suitable $a(i) \in X$, hence f is bounded by $M_{a(i)} > 0$ on the neighborhood U_i.

   Again by paracompactness, there exists a partition of unity $\{\eta_i \mid i \in I\}$, with $\operatorname{supp}\eta_i \subset U_i$ for all $i \in I$. The function $M_{a(i)}\, \eta_i$ is continuous and has support contained in U_i.

Second reasoning. By paracompactness, there exists a partition of unity $\{\eta_i \mid i \in I\}$ which is subordinated to $\mathscr{V}$. Thus, for every $i \in I$ there exists a $V_{a(i)} \in \mathscr{V}$ such that $\operatorname{supp}\eta_i \subset V_{a(i)}$. It follows that the function f is on $V_{a(i)}$ bounded by a constant $M_{a(i)} > 0$. The function $M_{a(i)}\,\eta_i$ is continuous and has support contained in $\operatorname{supp}(\eta_i)$.

From both reasonings given above, it follows that for all i, we have $f\eta_i \leq M_{a(i)}\eta_i$ on $\operatorname{supp}(\eta_i)$ hence on X. Furthermore, the sum $g := \sum_i M_{a(i)}\,\eta_i$ is a locally finite sum of continuous functions, hence continuous.

Finally, for $x \in X$ we have

$$f(x) = \sum_{i \in I} f(x)\eta_i(x) \leq \sum_{i \in I} M_{a(i)}\eta_i(x) = g(x).$$

Solution to 4

(a) For $\gamma_1, \gamma_2 \in \Gamma$ we have

$$\rho_{\gamma_1\gamma_2} = (\alpha_{\gamma_1\gamma_2}, \beta_{\gamma_1\gamma_2}) = (\alpha_{\gamma_1}\alpha_{\gamma_2}, \beta_{\gamma_1}\beta_{\gamma_2}) = \rho_{\gamma_1}\rho_{\gamma_2},$$

and $\rho_1 = (\alpha_1, \beta_1) = (\mathrm{id}_{S^1}, \mathrm{id}_{S^1}) = \mathrm{id}_{S^1 \times S^1}$. Therefore, ρ defines an action of Γ on $S^1 \times S^1$. For a given γ the maps $\alpha_\gamma, \beta_\gamma : S^1 \to S^1$ are continuous, hence so is $\rho_\gamma = (\alpha_\gamma, \beta_\gamma) : S^1 \times S^1 \to S^1 \times S^1$. It follows that ρ is an action by homeomorphisms on $S^1 \times S^1$.

(b) Let $p = (x, y)$. Then the orbit Γp consists of $1p = p = (x, y)$ and $gp = (-x, -y_1, y_2)$. Since $x \neq -x$, each orbit consists of precisely two points.

(c) It is obvious that f is continuous. We claim that f is injective. Indeed, let $f(s, y) = f(s', y')$, for $(s, y), (s', y') \in [0, 1] \times S^1$. Then $y = y'$ and $\cos s\pi = \cos s'\pi$ and $\sin s\pi = \sin s'\pi$. Since $\pi s \in [0, \pi]$, the latter two conditions imply that $s = s'$. Hence, f is injective. Finally, since $[0, 1] \times S^1$ is compact and $S^1 \times S^1$ Hausdorff, it follows that f is a topological embedding.

(d) Let $z \in S^1 \times S^1/\Gamma$ and select $(x, y) \in S^1 \times S^1$ such that $\pi(x, y) = z$. We note that $x = (\cos \pi s, \sin \pi s)$ for a unique $s \in [0, 2)$.

If $s \in [0, 1]$ then $\pi(x, y) = F(s, y)$ and we are done.

If $s > 1$, then $-x = (\cos \pi(s-1), \sin \pi(s-1))$ hence

$$\pi(x, y) = \pi(g(x, y)) = \pi(-x, \beta_g y) = \pi f(s-1, \beta_g y) = F(s-1, \beta_g y).$$

Since $(s-1, y) \in [0, 1] \times S^1$, we see that F is surjective.

(e) We observe that the map F induces an injective map $\bar{F} : [0, 1] \times S^1/\sim \to S^1 \times S^1/\Gamma$ such that $\bar{F} \circ \mathrm{pr} = F$. Here $\mathrm{pr} : [0, 1] \times S^1 \to [0, 1] \times S^1/\sim$ is the canonical projection. We claim that $\bar{F}$ is a homeomorphism from $[0,1]\times S^1/\sim$ onto $S^1\times S^1/\Gamma$. Since F is surjective, $\bar{F}$ is bijective.

Now $S^1\times S^1/\Gamma$ is the quotient of a Hausdorff space by a finite group action, hence a Hausdorff space. Since $[0,1]\times S^1$ is the product of two compact spaces, it is compact. Therefore, the bijective continuous map $\bar{F}:[0,1]\times S^1/\sim\ \to S^1\times S^1/\Gamma$ is a homeomorphism.

(f) Since

$$F(1,y)=\pi(-1,0,y)=\pi(1,0,\beta_g y)=F(0,\beta_g y) \qquad (*)$$

we see that the surjectivity of F implies that F maps $[0,1)\times S^1$ onto $S^1\times S^1/\Gamma$. We will now show that F is injective on $[0,1)\times S^1$. If $s,s'\in[0,1)$, $y,y'\in S^1$ and $F(s,y)=F(s',y')$ then it follows that $(\cos\pi s',\sin\pi s',y')=\gamma(\cos\pi s,\sin\pi s,y)$ with either $\gamma=1$ or $\gamma=g$. Assume the latter. Then

$$(\cos\pi s',\sin\pi s')=\alpha_g(\cos\pi s,\sin\pi s)=(-\cos\pi s,-\sin\pi s).$$

Since $\sin\pi s\geq 0$ and $\sin\pi s'\geq 0$ this implies $\sin\pi s=\sin\pi s'=0$ hence $s=0=s'$ and then $\cos\pi s'=1=\cos\pi s$, contradiction.

We thus see that $\gamma=1$ hence $(\cos\pi s',\sin\pi s',y')=(\cos\pi s,\sin\pi s,y)$. Hence, $s=s'$ and $y=y'$ and the injectivity follows.

(g) We will now describe the fibers of F. From (*) we obtain that

$$F(1,y)=F(0,\beta_g y).$$

so that the fiber of $F(1,y)$ contains $(0,\beta_g y)$ and $(1,y)$. Since F is injective on $[0,1)\times S^1$, we see that the fiber of $F(1,y)$ cannot contain any other point. Again by injectivity of F on $[0,1)$ it follows that the fiber of $F(s,y)$ for $s\notin\{0,1\}$ can only contain the point (s,y).

We see that for two distinct points $(s,y),(s',y')$ with $s\leq s'$ we have $(s,y)\sim(s',y')$ if and only if $s=0$, $s'=1$ and $y'=\beta_g y=(-y_1,y_2)$.

From this it is clear that $[0,1]\times S^1/\sim$ equipped with the quotient topology is homeomorphic to the Klein bottle.

In particular, it follows that $S^1\times S^1/\Gamma$ is homeomorphic to the Klein bottle.