Solutions for the inleiding topologie exam

Lennart Meier

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Below you find the solutions. What I write here is rather short, but contains all the crucial points. I apologize for the lack of graphics.

Problem 1 (5 points)**.** *Consider the equivalence relation $\sim$ on $\mathbb{R}$, where $x \sim y$ iff x and y are both positive or both negative or both zero. Let $X = \mathbb{R}/\sim$ with the quotient topology. List (with proof) all closed subsets of X.*

The space X has only three points, which we call $-$, 0 and $+$. The closed sets are $\varnothing$, $\{0\}$, $\{-, 0\}$, $\{0, +\}$ and X. Indeed, by definition of the quotient topology a set in X is open if and only if its preimage in $\mathbb{R}$ is open. Thus our claim is that the preimage of $A \subset X$ in $\mathbb{R}$ is open if and only if A is X, $\{+, -\}$, $\{+\}$, $\{-\}$ or $\varnothing$. This is easy to check in each single case, e.g. the preimage of $\{-\}$ is $(-\infty, 0)$, which is open and the preimage of $\{0, +\}$ is $[0, \infty)$, which is not open.

Problem 2 (10 points)**.** *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $f\colon X \to Y$ a function. Let further $(C_i)_{i\in I}$ be a cover of X by closed subsets $C_i \subseteq X$, i.e. $\bigcup_{i\in I} C_i = X$. Assume that $f|_{C_i}\colon C_i \to Y$ is a continuous function for each $i \in I$, where we equip C_i with the subspace topology.*

(a) Give an example of X, Y, f and $(C_i)_{i\in I}$ as above, where f is not *continuous.*

(b) Show that f is necessarily continuous if the indexing set I is finite.

For part a: Let $f\colon X \to Y$ be any non-continuous function, $I = X$ and $C_i = \{i\}$. As every function from a point is continuous, this fulfills the conditions. E.g. take $f\colon \mathbb{R} \to \mathbb{R}$, with $f(x) = 0$ if $x \leq 0$ and 1 else.

For part b: We have seen that a function $f\colon X \to Y$ is continuous if and only if $f^{-1}(A)$ is closed for every closed subset $A \subset Y$. Let $A \subset Y$ be closed. Then $P_i = (f|_{C_i})^{-1}(A)$ is closed in C_i and thus closed in X. (Indeed, by definition there exists an open $U_i \subset X$ with $U_i \cap C_i = C_i \setminus P_i$. Thus, $(X \setminus U_i) \cap C_i = P_i$, which is thus the intersection of two closed subsets and thus closed.) Clearly, $f^{-1}(A) = \bigcup_{i\in I} P_i$, which is closed since *finite* unions of closed subsets are closed.

Problem 3 (6 points)**.** *Let $X = \mathcal{C}([0, 1], \mathbb{R})$ be the set of all continuous functions from $[0, 1]$ to $\mathbb{R}$. We consider on it the topology induced by the metric $d(f, g) = \max_{x\in[0,1]} |f(x) - g(x)|$. Show that X is path-connected.*

Let $f, g \in X$. We want to show that they can be connected by a path. We assume $f \neq g$ as the other case is obvious. Define $\gamma\colon [0,1] \to X$ by $\gamma(t)\colon s \mapsto tf(s) + (1-t)g(s)$. It remains to show that γ is continuous. We have

$$\begin{aligned} |t_0 f(s) + (1-t_0)g(s) - t_1 f(s) - (1-t_1)g(s)| &= |(t_0 - t_1)f(s) + (t_1 - t_0)g(s)| \\ &= |t_0 - t_1||f(s) - g(s)| \end{aligned}$$

Thus $d(\gamma(t_0), \gamma(t_1)) = |t_0 - t_1| d(f, g)$. We see that for $\varepsilon > 0$ and $|t_0 - t_1| < \delta = \frac{\varepsilon}{d(f,g)}$, we obtain indeed $d(\gamma(t_0), \gamma(t_1)) < \varepsilon$.

Problem 4 (6 points)**.** *Give examples $i, j\colon S^1 \to T$ of embeddings of the circle into the torus such that $T \setminus i(S^1)$ is not homeomorphic to $T \setminus j(S^1)$.*

We view T as the quotient of the unit square $Q = [0,1]^2$ by the usual equivalence relation with quotient map $p\colon Q \to T$. Let $i'\colon S^1 \to Q$ the embedding as the circle of radius $\frac{1}{4}$ around $(\frac{1}{2}, \frac{1}{2})$ and $i = pi'$. This is clearly injective and hence an embedding (as we are between compact Hausdorff spaces). Let $j\colon S^1 \to T$ be induced by the map

$$[0,1] \to Q, \qquad t \mapsto (t, 0).$$

This is an embedding for the same reason.

We claim that $T \setminus i(S^1)$ is disconnected. Indeed, p is closed (since it is a map between compact Hausdorff spaces) and thus also open since it is surjective. Let U be the open ball of radius $\frac{1}{4}$ around $(\frac{1}{2}, \frac{1}{2})$ and let V be the complement of its closure in Q. Then $p(U)$ and $p(V)$ are non-empty open disjoint subsets of T that cover $T \setminus i(S^1)$.

We claim that $T \setminus j(S^1)$ is connected. Indeed, it is a quotient of $(0,1) \times [0,1]$, which is connected.

Thus the two complements cannot be homeomorphic.

Problem 5 (10 points)**.** *(a) Consider the open cover $\mathcal{U} = \{(-2,1), (-1,2)\}$ of the interval $(-2,2)$ with the Euclidean topology. Give an example of a partition of unity subordinate to $\mathcal{U}$.*

(b) Let $X = \{x, y\}$ be a set with two elements. Give an example of a topology on X such that every open cover[1] has a subordinate partition of unity, but the topology is not Hausdorff.

Part a: Let $\eta_1(x) = \begin{cases} 1 & \text{if } x(-2, -\frac{1}{2}) \\ \frac{1}{2} - x & \text{if } x \in [-\frac{1}{2}, \frac{1}{2}] \\ 0 & \text{if } x \in (\frac{1}{2}, 2) \end{cases}.$

Set $\eta_2 = 1 - \eta_1$. The supports of η_1 and η_2 are $(-2, \frac{1}{2}]$ and $[-\frac{1}{2}, 2)$.

Part b: We take the indiscrete topology on X. This is not Hausdorff as the only neighborhood of x is X, containing y. The two possible open covers are $\{X\}$ and $\{X, \varnothing\}$. In the first case the function $\eta = 1$ defines a subordinate partition of unity. In the second case take $\eta_1 = 1$ and $\eta_2 = 0$.

[1] For simplicity, you are allowed to assume that no open set occurs twice in the open cover.

Problem 6 (13 points)**.** *Let $p, q \in S^2$ be the points $(1,0,0)$ and $(-1,0,0)$. Let $X = S^2/\{p,q\}$ with the quotient topology.*

(a) Show that X is not a 2-dimensional manifold.

(b) Show that $S^2 \setminus \{p,q\}$ is not homeomorphic to $\mathbb{R}^2$. (Hint: Use one-point compactifications.)

Part a: Denote by $[p] \in X$ the image of $p \in S^2$. Note that the restriction of the quotient map $\pi\colon S^2 \to X$ defines a homeomorphism $S^2 \setminus \{p,q\} \to X \setminus \{[p]\}$.

Suppose that X is a 2-dimensional manifold. Then there exists an open neighborhood U around $[p]$ together with a homeomorphism $\varphi\colon U \to \mathbb{R}^2$. Set $U_n = \varphi^{-1}(B_d(\varphi([p]), \frac{1}{n}))$. Then $U_n \setminus \{[p]\}$ is path-connected. Indeed: this is homeomorphic to $B_d(0, \frac{1}{n}) \setminus \{0\}$. Every point in it can be connected by a straight line with $(0, \frac{1}{2n})$ or with $(0, -\frac{1}{2n})$ and these two can be connected by a semicircle.

Let $V_+ = \{(x,y,z) \in S^2 : x > 0\}$ and $V_- = \{(x,y,z) \in S^2 : x < 0\}$. Since the set $\pi(V_+ \cup V_-)$ is open, we must have $\pi^{-1}(U_n) \subset V_+ \cup V_-$ for some n. Since $\pi^{-1}(U_n)$ is an open neighborhood of $\{p,q\}$, it must intersect both V_+ and V_- non-trivially. But then $\pi^{-1}(U_n \setminus \{[p]\}) \cap V_+$ and $\pi^{-1}(U_n \setminus \{[p]\}) \cap V_-$ form a cover by non-empty open disjoint sets, showing that $\pi^{-1}(U_n \setminus \{[p]\})$ and hence $U_n \setminus \{[p]\}$ is disconnected, in contradiction with what we showed above.

Part b: Suppose that $S^2 \setminus \{p,q\}$ and $\mathbb{R}^2$ are homeomorphic. Then also their one-point compactifications A and B are homeomorphic. We have seen in class that $B \cong S^2$, hence it is a 2-dimensional manifold. We claim that $A \cong S^2/\{p,q\}$. Using part a we see that A is not a 2-dimensional manifold and thus A and B cannot be homeomorphic.

Our claim follows by checking the following points:

- $X = S^2/\{p,q\}$ is compact (as a quotient of a compact space)
- X is Hausdorff: since S^2 is Hausdorff and normal, we can for any $x \neq p, q$ find disjoint open neighborhoods U around x and V around $\{p,q\}$; their images are disjoint open neighborhoods around $[x]$ and $[p]$. Separating two point $x \neq y$ in X that are not $[p]$ is even easier.
- $X \setminus \{[p]\}$ is homeomorphic to $S^2 \setminus \{p,q\}$

The one-point compactification is uniquely determined by these properties. (Cf. Theorem 4.40)