# RETAKE COMPLEX FUNCTIONS

JULY 21 2015, 9:00-12:00

- Put your name and student number on every sheet you hand in.
- When you use a theorem, show that the conditions are met.
- Include your partial solutions, even if you were unable to complete an exercise.

**Exercise 1 (*10 pt*):** Let $\gamma$ be a closed path in $\mathbb{C}$ that winds $k \in \mathbb{Z}$ times around $z_0 \in \mathbb{C}$. Let $f(z) = (z - z_0)^n$, where $n \in \mathbb{N}$. How many times does $f \circ \gamma$ (i.e. the image of $\gamma$ under $f$) wind around 0 ?

**Exercise 2 (*20 pt*):** Prove the following *Cauchy's bound*: Every complex root of the algebraic equation

$$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0 \qquad (a_j \in \mathbb{C})$$

satisfies

$$|z| < 1 + \max\{|a_0|, |a_1|, \cdots, |a_{n-1}|\}.$$

**Exercise 3 (*20 pt*):** Prove that *Joukowski's function*

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

provides an analytic isomorphism between the open upper half-plane $H \subset \mathbb{C}$ and the set $U = \{z \in H : |z| > 1\}$.

**Exercise 4 (*30 pt*):** Let $n \geq 1$ be an integer number. Consider the complex function

$$f(z) = \frac{1}{(1 + z^2)^{n+1}}.$$

**a.** (*5 pt*) Find and classify all singular points of $f$ in $\mathbb{C}$.

**b.** (*15 pt*) Compute the residue of $f$ in each singular point.

**Turn the page!**

**c.** (*10 pt*) Prove that

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}}\, dx \;=\; \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\,\pi\ .$$

**Exercise 5 (*20 pt*):** Let $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$ be two analytic functions that have an equal (finite) number of zeros. Prove that there exist an analytic function $h : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ and a closed path $\gamma$, with the interior containing all zeros of $f$ and $g$, such that

$$|f(z) - h(z)g(z)| < |f(z)|$$

on $\gamma$.

**Bonus Exercise (*10 pt*):** Prove that

$$\int_{C_2} \frac{1}{z-1} \sin\left(\frac{1}{z}\right) dz = 0.$$

where $C_2$ is the circle of radius 2 with center $z = 0$, that is traced counter-clockwise once.