RETAKE COMPLEX FUNCTIONS

JULY 18, 2017, 9:00-12:00

- Put your name and student number on every sheet you hand in.
- When you use a theorem, show that the conditions are met.
- Include your partial solutions, even if you were unable to complete an exercise.

Notation: For $a \in \mathbb{C}$ and $r > 0$, we write $D(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$, and $\overline{D}(a,r)$ and $C(a,r)$ are the closure and boundary respectively of $D(a,r)$.

Exercise 1 (*15 pt*):

(a) Determine the image $f(\mathbb{C})$ of the exponential function $f(z) = e^z$.

A theorem of Picard states that for a non-constant entire function $f\colon \mathbb{C} \to \mathbb{C}$ the following holds: the complement in $\mathbb{C}$ of the image $f(\mathbb{C})$ is either empty or consists of exactly one point. You may use this result in the rest of this exercise.

(b) Let $g\colon \mathbb{C} \to \mathbb{C}$ be an injective entire function. Prove that g is surjective. *Hint: Consider the function $z \mapsto g(e^z)$. Alternatively, consider the function $z \mapsto g(g(z))$.*

(c) Explain carefully why g is necessarily an analytic automorphism of $\mathbb{C}$.

Exercise 2 (*15 pt*):

Let $U = D(0,r) \setminus \{0\}$ be a punctured open disc and let $f\colon U \to \mathbb{C}$ be an analytic function that is injective.

(a) Assume that the isolated singularity of f at 0 is removable. Let $g\colon D(0,r) \to \mathbb{C}$ be the analytic extension of f to $D(0,r)$. Show that the order of g at 0 is either 0 or 1.

(b) Assume instead that f has a pole at 0. Show that this is necessarily a simple pole.

(c) Prove that f cannot have an essential singularity at 0. (In other words, either (a) or (b) must occur.)

Exercise 3 (*15 pt*):

Prove that the following integral converges and evaluate it.

$$\int_0^\infty \frac{(\log x)^2}{x^4+1}\,dx.$$

(*Hint: Use a contour consisting of two circular arcs and two segments. Use an appropriate definition of the complex logarithm.*)

Exercise 4 (*15 pt*):

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence r, with $0 < r \leq \infty$. For $m \in \mathbb{Z}_{\geq 0}$, let p_m be the polynomial

$$p_m(z) = \sum_{n=0}^{m} a_n z^n.$$

Let w be a zero of f with $|w| < r$. Prove that for all $\varepsilon > 0$, there exists $N \in \mathbb{Z}_{\geq 0}$ such that for all $m \geq N$, the function p_m has a zero in $D(w, \varepsilon)$.

Exercise 5 (*15 pt*): Let

$$f(z) = \frac{(z^4 - 1)^2 \sin^2 z}{\cos 2\pi z - 1}$$

and let $U \subset \mathbb{C}$ be the domain of f. Let $V \subset \mathbb{C}$ be the maximal open set on which a holomorphic function g can be defined that agrees with f on U. Determine the radius of convergence of the power series for $g(z)$ at each of the following points:

$$z = i, \qquad z = 1 + i, \qquad z = 2 + i, \qquad z = 3 + i.$$

Exercise 6 (*15 pt*):

In this exercise, you may freely use the fact from real analysis that for all $a \in \mathbb{R}_{>1}$, we have

$$\int_1^{\infty} x^{-a} dx < \sum_{n=1}^{\infty} n^{-a} < 1 + \int_1^{\infty} x^{-a} dx \tag{1}$$

(which you can easily see by considering upper and lower Riemann sums of the integral).

(a) Show that the series

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$$

(with $n^{-z} := e^{-z \log n}$) defines a holomorphic function on $U := \{z \in \mathbb{C} : \mathrm{Re}(z) > 1\}$.

One can show that ζ has an analytic continuation to $\mathbb{C} \backslash \{1\}$. This function $\zeta \colon \mathbb{C} \backslash \{1\} \to \mathbb{C}$ is meromorphic; it is called the Riemann zeta-function. You may use this in the rest of this exercise.

(b) Show that ζ has a simple pole in 1, with residue 1.