Complex analysis – Exam

Notes:

1. **Write your name and student number *****clearly***** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books, the lecture's slides and your own notes.
5. You are **not** allowed to consult colleagues, calculators, or use the internet to assist you solve exam questions.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

Exercise 1 (1.0 pt)**.** For a real number x, we know that $\sin^2 x + \cos^2 x = 1$. If we extend sin and cos to the complex plane using their respective power series expansions, does it still hold that $\sin^2 z + \cos^2 z = 1$ for all complex numbers?

Proof 1. Both sin and cos are, by the command of the question, analytic functions. One readily sees that $\lim |a_n|^{1/n} = 0$ for both of these series, so they have infinite radius of convergence (you did not have to argues this though, everybody knows these series have infinite radius of convergence). Therefore $\sin^1 z + \cos^2 z$ is an analytic function on $\mathbb{C}$ and so is 1. These two analytic functions agree on a set with accumulation point, namely, the real line, hence they agree in the the whole complex plane and the identity $\sin^2 z + \cos^2 z = 1$ holds for all complex numbers □

Proof 2. From the powe series expansion for sin and cos we see that

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Therefore

$$\begin{aligned} \sin^2 z + \cos^2 z &= -\frac{e^{2iz} - 2 + e^{-2iz}}{4} + \frac{e^{2iz} + 2 + e^{-2iz}}{4} \\ &= \frac{4}{4} = 1 \end{aligned}$$

□

Exercise 2 (1.0 pt)**.** Let $f, g\colon \mathbb{C} \to \mathbb{C}$ be holomorphic functions such that $|f(z)| \leq |g(z)|$ for all z. Show that there is a complex number λ such that $f = \lambda g$.

Proof. If g is identically 0, then so is f and we can take $\lambda = 1$.

If g is not identically 0, then $h := f/g$ is a holomorphic function with singularities at the zeros of g. Since $|f| \leq |g|$, we have that $|h| \leq 1$ and in particular h is bounded in a neighbouhoord or each of its singularities. So all singularities of h are removable and we can extend h to an entire function.

Since h is an entire function which satisfies $|h| \leq 1$, (it is therefore bounded) by Liouville's theorem h is constant, say $h \equiv \lambda$. Therefore $f = \lambda g$. □

Exercise 3. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a function which is bounded by $\log |z|$ for $|z|$ large, that is, there is $C > 1$ such that if $|z| > C$, then $|f(z)| \leq \log |z|$.

1. (1.0 pt) Show that if f is holomorphic, then f is constant.

2. (1.0 pt) If f is harmonic, does it have to be constant?

Proof.

- Consider $g = e^f$. Then for z large enough

$$|g| = |e^f| \leq e^|f| \leq e^{\log |z|} = |z|,$$

showing that g is a nowhere vanishing holomorphic function (it is an exponential of something) bounded by $|z|$. From a hand-in exercise which is also a result in the book, g is a polynomial of degree at most 1, say $g(z) = az + b$, which never vanishes, hence g is constant and so is f.

- For this part we can deal with the real and imaginary parts of f separately, since if we can show that they are both constant, then so is f. So from now we assume that f takes real values. In this case, since $\mathbb{C}$ is simply-connected, there is a holomorphic function $\tilde{f}$ whose real part is f and we can consider as before the function $g = e^{\tilde{f}}$. Then

$$|g| = |e^{\tilde{f}}| = |e^f| =\leq e^|f| \leq e^{\log |z|} = |z|,$$

showing one again that g is a linear polynomial without roots, and hence constant. Therefore f is also constant.

Side note: $\log |z|$ is not a counter example to the second question because it is not defined on the whole plane. For the same reason $f(z) = 1/z$ is not a counter example for the question. □

Exercise 4. Let $f\colon \mathbb{C} \to \mathbb{C}$ be the holomorphic function with singularities given by

$$f(z) = \frac{e^{iz}}{z^4 + 1}.$$

1. (0.7 pt) Determine the singularities of f and for each of them, determine what type of singularity it is (removable, pole or essential).

2. (0.7 pt) Express the residue of f at each of its singularities in terms of the 8^{th} root of 1, $\omega = e^{\frac{2\pi i}{8}}$.

3. (0.6 pt) Relate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + 1} dx$$

to the residues of f.

Proof. This is very similar to a question in the mock exam and I will not repeat its solution here. □

Exercise 5 (1.5 pt)**.** Let $f\colon \mathbb{C} \to \mathbb{C}$ be given by

$$f(z) = \frac{z^4}{1 + z + 2z^2 \cdots + 1000z^{1000}}.$$

Compute the sum of all residues of f.

Proof. The function f is a quotient of two polynomials, hence it is meromorphic and has finitely many poles (at most 1000). By Cauchy's integral formula we can compute the sum of the residues of f by integrating it along any large circle which contains all poles of f inside it. So we have

$$\begin{aligned}
|2\pi i \sum_z \mathrm{Res}_z f| &= \left| \int_{\partial B_R} \frac{z^4}{1 + z + 2z^2 \cdots + 1000z^{1000}} dz \right| \\
&= \int_{\partial B_R} \left| \frac{z^4}{1 + z + 2z^2 \cdots + 1000z^{1000}} \right| |dz| \\
&\le C \int_{\partial B_R} \frac{R^4}{R^{1000}} |dz| \\
&\le C 2\pi R \frac{1}{R^{996}} \\
&\le C 2\pi \frac{1}{R^{995}} \xrightarrow{R \to \infty} 0.
\end{aligned}$$

Therefore $\sum_z \mathrm{Res}_z f = 0$. ∎

Exercise 6 (1.5 pt)**.** Let a be a real number bigger that 1. Show that the equation

$$e^z - z^n e^a = 0$$

has n solutions inside the unit disc (counted with multiplicity).

Proof. We will use Rouché's theorem. Take $f(z) = -z^n e^a$ and $g(z) = e^z - z^n e^a$, then for z in the unit circle,

$$|f(z) - g(z)| = | - z^n e^a - e^z + z^n e^a| = |e^z| \le e < e^a = |f(z)|,$$

hence g and f have the same number of zeros inside the unit disc (counted with multiplicity). It is clear that f has a zero of order n at the origin, so g also has n zeros inside the disc. ∎

Exercise 7.

1. (0.5) Show that the first quadrant

$$R = \{(x + iy) \in \mathbb{C} \colon x > 0, \text{ and } y > 0\}$$

is isomorphic to the upper half plane

$$H = \{(x + iy) \in \mathbb{C} \colon y > 0\}.$$

2. (0.5) Determine all the automorphisms of R.

Proof.

- The function $f \colon R \to H$, $f(z) = z^2$ is a biholomorphism between R and H whose inverse is given by

$$g(z) = e^{\frac{1}{2}\log z} =: \sqrt{z},$$

where we use the principal branch of the log above (and you should argue that these are indeed inverses of each other)

- Since the automorphisms of H are the fractional linear transformations of the form

$$\varphi(z) = \frac{az + b}{cz + d} \qquad \text{with } a, b, c, d \in \mathbb{R}, ad - bc > 0,$$

we have that the automorphisms of R are of the form $f^{-1} \circ \varphi \circ f$, which explicitly becomes

$$z \mapsto \sqrt{\frac{az^2 + b}{cz^2 + d}} \qquad \text{with } a, b, c, d \in \mathbb{R}, ad - bc > 0.$$

∎