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Measure and Integration: Solutions Quiz 2012-13

1. Consider the measure space $([0,1),\mathcal{B}([0,1)),\lambda)$, where $\mathcal{B}([0,1))$ is the Borel σ-algebra restricted to $[0,1)$ and λ is the restriction of Lebesgue measure on $[0,1)$. Define the transformation $T:[0,1)\to[0,1)$ given by

$$T(x) = \begin{cases} 3x & 0\le x<1/3, \\ 3x-1, & 1/3\le x<2/3 \\ 3x-2, & 2/3\le x<1. \end{cases}$$

   (a) Show that T is $\mathcal{B}([0,1))/\mathcal{B}([0,1))$ measurable.

   (b) Determine the image measure $T(\lambda)=\lambda\circ T^{-1}$.

   (c) Let $\mathcal{C}=\{A\in\mathcal{B}([0,1)) : T^{-1}A=A\}$. Show that $\mathcal{C}$ is a σ-algebra.

Solution(a): To show T is $\mathcal{B}([0,1))/\mathcal{B}([0,1))$ measurable, it is enough to consider inverse images of intervals of the form $[a,b)\subset[0,1)$. Now,

$$T^{-1}([a,b)) = [\frac{a}{3},\frac{b}{3})\cup[\frac{a+1}{3},\frac{b+1}{3})\cup[\frac{a+2}{3},\frac{b+2}{3})\in\mathcal{B}([0,1)).$$

Thus, T is measurable.

Solution(b): We claim that $T(\lambda)=\lambda$. To prove this, we use Theorem 5.7. Notice that $\mathcal{B}([0,1))$ is generated by the collection $\mathcal{G}=\{[a,b):0\le a\le b<1\}$ which is closed under finite intersections. Now,

$$\begin{aligned} T(\lambda)([a,b)) &= \lambda(T^{-1}([a,b))) \\ &= \lambda([\frac{a}{3},\frac{b}{3}))+\lambda([\frac{a+1}{3},\frac{b+1}{3}))+\lambda([\frac{a+2}{3},\frac{b+2}{3})) \\ &= b-a=\lambda([a,b)). \end{aligned}$$

Since the constant sequence $([0,1))$ is exhausting, belongs to $\mathcal{G}$ and $\lambda([0,1)) = T(\lambda([0,1))=1<\infty$, we have by Theorem 5.7 that $T(\lambda)=\lambda$.

Solution(c): We check the three conditions for a collection of sets to be a σ-algebra. Firstly, the empty set $\emptyset\in\mathcal{B}([0,1))$ and $T^{-1}(\emptyset)=\emptyset$, hence $\emptyset\in\mathcal{C}$. Secondly, Let $A\in\mathcal{C}$, then $T^{-1}A=A$. Now,

$$T^{-1}(X\setminus A)=T^{-1}X\setminus T^{-1}A=X\setminus T^{-1}A=X\setminus A.$$

Thus, $X \setminus A \in \mathcal{B}([0,1))$ and $T^{-1}(X \setminus A) = X \setminus A$. This implies $X \setminus A \in \mathcal{C}$. Thirdly, let (A_n) be a sequence in $\mathcal{C}$, then $A_n \in \mathcal{B}([0,1))$ and $T^{-1}A_n = A_n$ for each n. Since $\mathcal{B}([0,1))$ is a σ-algebra, we have $\bigcup_n A_n \in \mathcal{B}([0,1))$, and

$$T^{-1}(\bigcup_n A_n) = \bigcup_n T^{-1}A_n = \bigcup_n A_n.$$

Thus, $\bigcup_n A_n \in \mathcal{C}$. This shows that $\mathcal{C}$ is a σ-algebra.

2. Let $\mathcal{B}(\mathbb{R}^n)$ be the Borel σ-algebra over $\mathbb{R}^n$, and let $\overline{\mathcal{B}}(\mathbb{R}^n)$ be the completion of $\mathcal{B}(\mathbb{R}^n)$ (In the notation of exercise 4.13, p.29, if $\mathcal{A} = \mathcal{B}(\mathbb{R}^n)$, then $\mathcal{A}^* = \overline{\mathcal{B}}(\mathbb{R}^n)$). The σ-algebra $\overline{\mathcal{B}}(\mathbb{R}^n)$ is called the Lebesgue σ-algebra over $\mathbb{R}^n$. Let $n = 1$ and suppose $M \subset \mathbb{R}$ is a **non**-Lebesgue measurable set (i.e. $M \notin \overline{\mathcal{B}}(\mathbb{R})$). Define $A = \{(x,x) \in \mathbb{R}^2 : x \in M\}$, and let $g : \mathbb{R} \to \mathbb{R}^2$ be given by $g(x) = (x,x)$.

   (a) Show that $A \in \overline{\mathcal{B}}(\mathbb{R}^2)$ i.e. A is Lebesgue measurable.

   (b) Show that g is a Borel measurable function, i.e. $g^{-1}(B) \in \mathcal{B}(\mathbb{R})$ for each $B \in \mathcal{B}(\mathbb{R}^2)$.

   (c) Show that $A \notin \mathcal{B}(\mathbb{R}^2)$, i.e. A is not Borel measurable.

   Solution(a): The set A is a subset of the diagonal line $L = \{(x,x) : x \in \mathbb{R}\}$ which is the image of the hyperplane $x = 0$ (the y-axis) under a rotation by 45^o which is a linear transformation. Hence by exercise 6.4 and Theorem 7.9 we have that $L \in \mathcal{B}(\mathbb{R}^2)$ and $\lambda^2(L) = 0$. Thus, A is a subset of a Borel set of measure zero, by exercise 4.13(iv) we have $A \in \overline{\mathcal{B}}(\mathbb{R}^2)$.

   Solution(b): This follows from the simple fact that g is a continuous function and hence $\mathcal{B}(\mathbb{R})/\overline{\mathcal{B}}(\mathbb{R}^2)$ measurable (see example 7.3).

   Solution(c): We give a proof by contradiction. Suppose that $A \in \mathcal{B}(\mathbb{R}^2)$, since G is measurable, then $g^{-1}(A) \in \mathcal{B}(\mathbb{R})$. However, $T^{-1}(A) = M$ and $M \notin \mathcal{B}(\mathbb{R})$, leading to a contradiction. Thus, $A \notin \mathcal{B}(\mathbb{R}^2)$.

3. Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra over $\mathbb{R}$, and λ is Lebesgue measure. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \sum_{k=0}^{2^n-1} \frac{k}{2^n} \cdot 1_{[k/2^n,(k+1)/2^n)},\ n \geq 1.$$

   (a) Show that f_n is measurable, and $f_n(x) \leq f_{n+1}(x)$ for all $x \in X$.

   (b) Let $f(x) = x\mathbf{1}_{[0,1)}(x)$. Show that f is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ measurable.

   (c) Prove that $f(x) = \lim_{n\to\infty} f_n(x) = \sup_{n\geq 1} f_n(x)$ for all $x \in \mathbb{R}$.

   (d) Show that $\int f d\lambda = \frac{1}{2}$.

   Solution(a): Since $[k/2^n, (k+1)/2^n) \in \mathcal{B}(\mathbb{R})$, then $1_{[k/2^n,(k+1)/2^n)}$ is a measurable function. Thus f_n is a linear combination of measurable functions (in fact f_n is a simple function) and hence measurable. For $x \notin [0,1)$, we have $f_n(x) = f_{n+1}(x) = 0$. Suppose $x \in [0,1)$, then there exists a $0 \le k \le 2^n - 1$ such that $x \in [k/2^n, (k+1)/2^n)$. Since

$$[k/2^n, (k+1)/2^n) = [2k/2^{n+1}, (2k+1)/2^{n+1}) \cup [(2k+1)/2^{n+1}, (2k+2)/2^{n+1}),$$

we see that $f_n(x) = \frac{k}{2^n}$ while $f_{n+1}(x) \in \{\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\}$ so that $f_n(x) \le f_{n+1}(x)$.

Solution(b): We consider inverse images of interval of the form $[a, \infty)$. Now,

$$f^{-1}([a,\infty)) = \begin{cases} \mathbb{R} & a \le 0, \\ [a,1), & 0 < a < 1 \\ \emptyset, & a \ge 1. \end{cases}$$

In all cases we see that $f^{-1}([a,\infty)) \in \mathcal{B}(\mathbb{R})$. Thus, f is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ measurable.

Solution(c): For $x \notin [0,1)$, we have $f(x) = f_n(x) = 0$ for all n. For $x \in [0,1)$, there exists for each n, an integer $k_n \in \{0, 1, \cdots, 2^n - 1\}$ such that $x \in [k_n/2^n, (k_n+1)/2^n)$. Thus,

$$|x - \frac{k_n}{2^n}| = |f(x) - f_n(x)| < \frac{1}{2^n}.$$

Since f_n is an increasing sequence, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \sup_n f_n(x).$$

Solution(d): We apply Beppo-Levi,

$$\begin{aligned} \int f \, d\lambda &= \lim_{n\to\infty} \int f_n \, d\lambda \\ &= \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \frac{k}{2^n} \lambda([k/2^n, (k+1)/2^n)) \\ &= \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \frac{k}{2^n} \frac{1}{2^n} \\ &= \lim_{n\to\infty} \frac{1}{4^n} \sum_{k=0}^{2^n-1} k \\ &= \lim_{n\to\infty} \frac{1}{2} \frac{(2^n-1)2^n}{4^n} = \frac{1}{2}. \end{aligned}$$

4. Let $(X, \mathcal{A}, \mu)$ be a measure space, and $u \in \mathcal{M}^+_{\overline{\mathbb{R}}}(\mathcal{A})$ satisfying $\int u \, d\mu < \infty$. For $a > 0$ (a real number) set $B_a = \{x \in X : u(x) > a\}$.

   (a) Show that for any $a > 0$ one has

   $$a\mathbf{1}_{B_a}(x) \le u(x) \text{ for all } x \in X.$$

   (b) Prove that $\mu(B_a) < \infty$ for all $a > 0$.

(c) Assume that $u(x) > 0$ for all $x \in X$, i.e. u is strictly positive. Show that μ is σ-finite, i.e. there exists an exhausting sequence $A_n \nearrow X$ with $\mu(A_n) < \infty$.

Solution(a): Since $u(x) \geq 0$ for all x, and for $x \in B_a$ one has $u(x) > a$, we get

$$a\mathbf{1}_{B_a}(x) \leq u(x)\mathbf{1}_{B_a}(x) \leq u(x).$$

for all $x \in X$ (note that if $x \notin B_a$, then the above inequalities reduce to $0 \leq u(x)$).

Solution(b):

$$\mu(B_a) = \int \mathbf{1}_{B_a}\, d\mu \leq \frac{1}{a} \int u\, d\mu < \infty.$$

Solution(c): Since $u(x) > 0$ for all $x \in X$, then

$$X = \bigcup_{n=1}^{\infty} \{x \in X : u(x) > \frac{1}{n}\} = \bigcup_{n=1}^{\infty} B_{\frac{1}{n}}.$$

Note that $(B_{\frac{1}{n}})$ is an increasing sequence and $\mu(B_{\frac{1}{n}}) < \infty$ (by part (b)). Thus, μ is σ-finite.