Universiteit Utrecht Universiteit Utrecht Boedapestlaan 6

Mathematisch Instituut 3584 CD Utrecht

Solutions Final Measure and Integration 2012-13

(1) Let $(E, \mathcal{B}, \nu)$ be a measure space, and $h : E \to \mathbb{R}$ a non-negative measurable function. Define a measure μ on $(E, \mathcal{B})$ by $\mu(A) = \int_A h d\nu$ for $A \in \mathcal{B}$. Show that for every non-negative measurable function $F : E \to \mathbb{R}$ one has

$$\int_E F\, d\mu = \int_E Fh\, d\nu.$$

Conclude that the result is still true for $F \in \mathcal{L}^1(\mu)$ which is not necessarily non-negative. (Hint: use a standard argument starting with indicator functions)

Proof Suppose first that $F = 1_A$ is the indicator function of some measurable set $A \in \mathcal{B}$. Then,

$$\int_E F\, d\mu = \mu(A) = \int_A h\, d\nu = \int_E 1_A h d\nu = \int_E F h d\nu.$$

Suppose now that $F = \sum_{k=1}^{n} \alpha_k 1_{A_k}$ is a non-negative measurable step function. Then,

$$\int_E F\, d\mu = \sum_{k=1}^{n} \alpha_k \mu(A_k) = \sum_{k=1}^{n} \alpha_k \int_E 1_A h d\nu = \int_E \sum_{k=1}^{n} \alpha_k 1_A h d\nu = \int_E F h d\nu.$$

Suppose that F is a non-negative measurable function, then there exists a sequence of non-negative measurable step functions F_n such that $F_n \uparrow F$. Then, $F_n h \uparrow Fh$, and by Beppo-Levi,

$$\int_E F\, d\mu = \lim_{n\to\infty} \int_E F_n\, d\mu = \lim_{n\to\infty} \int_E F_n h d\nu = \int_E F h d\nu.$$

Finally, suppose that $F \in \mathcal{L}^1(\mu)$. Since F^+, F^- are non-negative, we have

$$\int_E F^+\, d\mu = \int_E F^+ h\, d\nu \text{ and } \int_E F^-\, d\mu = \int_E F^- h\, d\nu.$$

Since $F \in \mathcal{L}^1(\mu)$, from the above we see that $Fh \in \mathcal{L}^1(\nu)$, hence

$$\int_E F\, d\mu = \int_E F^+\, d\mu - \int_E F^-\, d\mu = \int_E F^+ h\, d\nu - \int_E F^- h\, d\nu = \int_E F h\, d\nu.$$

(2) Consider the measure space $((0,\infty), \mathcal{B}((0,\infty), \lambda)$, where $\mathcal{B}((0,\infty))$ and λ are the restrictions of the Borel σ-algebra and Lebesgue measure to the interval $(0,\infty)$. Show that

$$\lim_{n\to\infty} \int_{(0,n)} \left(1 + \frac{x}{n}\right)^n e^{-2x}\, d\lambda(x) = 1.$$

(Hint: note that $1 + x \le e^x$).

Proof: Let $u_n(x) = \mathbf{1}_{(0,n)} \left(1 + \frac{x}{n}\right)^n e^{-2x}$, then $\lim_{n\to\infty} u_n(x) = \mathbf{1}_{(0,\infty)} e^{-x}$. Using the fact that $1+x \le e^x$, we see that $u_n(x) \le \mathbf{1}_{(0,\infty)} e^{-x}$. Since the function e^{-x} is positive, measurable and the improper Riemann integrable on $[0,\infty)$ exists, it follows that it is Lebesgue integrable on $[0,\infty)$ (and hence also on $(0,\infty)$). By Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \int_{(0,n)} \left(1 + \frac{x}{n}\right)^n e^{-2x}\, d\lambda(x) &= \lim_{n\to\infty} \int u_n(x) d\lambda(x) \\
&= \int \mathbf{1}_{(0,\infty)} e^{-x} d\lambda(x) = \int_0^\infty e^{-x}\, dx = 1.
\end{aligned}$$

(3) Let $(X, \mathcal{A}, \mu)$ be a probability space (i.e. $\mu(X) = 1$) and let $\{f_n\}$ be a sequence in $\mathcal{L}^1(\mu)$ such that $\int_X |f_n| d\mu = n$ for all $n \geq 1$. Let

$$A_n = \{x : |f_n(x) - \int_X f_n d\mu| \geq n^3\}.$$

(a) Show that $\mu\left(\bigcap_{m\geq 1}\bigcup_{n\geq m} A_n\right) = 0$. (Hint: use Exercise 6.9 (Borel-Cantelli Lemma)).
(b) Use part (a) to show that for every $\epsilon > 0$ there exists $m_0 \geq 1$ such that

$$\mu\{x \in X : |f_n(x)| < n^3 + n, \text{ for all } n \geq m_0\} > 1 - \epsilon.$$

Proof (a) By Markov Inequality we have

$$\mu(A_n) \leq \frac{1}{n^3}\int_X |f_n(x) - \int_X f_n d\mu|\, d\mu \leq \frac{2n}{n^3} = \frac{2}{n^2}.$$

Since $\displaystyle\sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \frac{2}{n^2} < \infty$, it follows by Borel-Cantelli Lemma (Exercise 6.9) that

$$\mu\left(\bigcap_{m\geq 1}\bigcup_{n\geq m} A_n\right) = 0.$$

Proof (b) By part (a) we have $\mu\left(\bigcup_{m\geq 1}\bigcap_{n\geq m} A_n^c\right) = 1$. By Theorem 4.4(iii),

$$\lim_{m\to\infty} \mu\left(\bigcap_{n\geq m} A_n^c\right) = \mu\left(\bigcup_{m\geq 1}\bigcap_{n\geq m} A_n^c\right) = 1.$$

Hence, given $\epsilon > 0$ there exists $m_0 \geq 1$ such that $\mu\left(\bigcap_{n\geq m_0} A_n^c\right) > 1 - \epsilon$. But for $x \in \bigcap_{n\geq m_0} A_n^c$ one has for $n \geq m_0$,

$$|f_n(x)| - |\int f_n\, d\mu| \leq |f_n(x) - \int f_n(x)\, d\mu| < n^3,$$

and thus, $|f_n(x)| < n^3 + n$. This implies that

$$\mu\{x \in X : |f_n(x)| < n^3 + n, \text{ for all } n \geq m_0\} \geq \mu\left(\bigcap_{n\geq m_0} A_n^c\right) > 1 - \epsilon.$$

(4) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and (A_i) a sequence in $\mathcal{A}$ such that $\lim_{n\to\infty} \mu(A_n) = 0$.
(a) Show that $\mathbf{1}_{A_n} \xrightarrow{\mu} 0$, i.e. the sequence $(\mathbf{1}_{A_n})$ converges to 0 in measure.
(b) Show that for any $u \in \mathcal{L}^1(\mu)$, one has $u\mathbf{1}_{A_n} \xrightarrow{\mu} 0$.
(c) Show that for any $u \in \mathcal{L}^1(\mu)$, one has

$$\sup_n \int_{\{|u|\mathbf{1}_{A_n} > |u|\}} |u|\mathbf{1}_{A_n}\, d\mu = 0.$$

(d) Show that $\lim_{n\to\infty} \int_{A_n} u\, d\mu = 0$.

Proof (a): For any $0 < \epsilon < 1$ and any$A \in \mathcal{A}$ with $\mu(A) < \infty$, we have

$$\mu(A \cap \{\mathbf{1}_{A_n} > \epsilon\}) = \mu(A \cap A_n) \leq \mu(A_n).$$

Thus, $\limsup_{n\to\infty} \mu(A \cap \{\mathbf{1}_{A_n} > \epsilon\}) = 0$ and hence $\lim_{n\to\infty} \mu(A \cap \{\mathbf{1}_{A_n} > \epsilon\}) = 0$. This implies $\mathbf{1}_{A_n} \xrightarrow{\mu} 0$.

Proof (b): Let $u \in \mathcal{L}^1(\mu)$. For any $\epsilon > 0$ and any$A \in \mathcal{A}$ with $\mu(A) < \infty$, one has

$$\mu(A \cap \{|u|\mathbf{1}_{A_n} > \epsilon\}) = \mu(A \cap A_n \cap \{|u| > \epsilon\}) \leq \mu(A_n).$$

This shows that $\lim_{n\to\infty} \mu(A \cap \{|u|\mathbf{1}_{A_n} > \epsilon\}) = 0$, and hence $u\mathbf{1}_{A_n} \xrightarrow{\mu} 0$.

Proof (c): Let $u \in \mathcal{L}^1(\mu)$. Note that $|u|\mathbf{1}_{A_n} \leq |u|$, thus the set $\{|u|\mathbf{1}_{A_n} > |u|\}$ is empty. By Theorem 10.9(ii), we have

$$\int_{\{|u|\mathbf{1}_{A_n} > |u|\}} |u|\mathbf{1}_{A_n}\, d\mu = 0$$

for all n and hence $\sup_j \int_{\{|u|\mathbf{1}_{A_n} > |u|\}} |u|\mathbf{1}_{A_n}\, d\mu = 0$.

Proof (d): By part (c) we see that the sequence $(|u|\mathbf{1}_{A_n})$ is uniformly integrable. Hence, by part (b) and Vitali's Theorem 16.6 we have

$$\lim_{n\to\infty} \int |u|\mathbf{1}_{A_n}\, d\mu = \lim_{n\to\infty} ||u\mathbf{1}_{A_n}||_1 = 0.$$

Since

$$\limsup_{n\to\infty} |\int u\mathbf{1}_{A_n}\, d\mu| \leq \limsup_{n\to\infty} \int |u|\mathbf{1}_{A_n}\, d\mu = \lim_{n\to\infty} \int |u|\mathbf{1}_{A_n}\, d\mu$$

the result follows.

(5) Let $E = \{(x, y) : 0 < x < \infty, 0 < y < 1\}$. We consider on E the restriction of the product Borel σ-algebra, and the restriction of the product Lebesgue measure $\lambda \times \lambda$. Let $f : E \to \mathbb{R}$ be given by $f(x, y) = y \sin x\, e^{-xy}$.
  (a) Show that f is $\lambda \times \lambda$ integrable on E.
  (b) Applying Fubini's Theorem to the function f, show that

$$\int_0^\infty \frac{\sin x}{x}\left(\frac{1 - e^{-x}}{x} - e^{-x}\right) dx = \frac{1}{2}\log 2.$$

Proof (a) Notice that f is continuous, and hence measurable. Furthermore, $|f(x, y)| \leq ye^{-xy}$. The fuction $g(x, y) = ye^{-xy}$ is non-negative measurable function, hence by Tonelli's Theorem,

$$\begin{aligned}\int_E |f(x, y)|d(\lambda \times \lambda)(x, y) &\leq \int_E ye^{-xy} d(\lambda \times \lambda)(x, y) \\ &= \int_0^1 \int_0^\infty ye^{-xy} dxdy \\ &= \int_0^1 1\, dy = 1.\end{aligned}$$

Notice that the integrands are Riemann integrable, hence the Riemann integral equals the Lebesgue integral, also the second equality is obtained by integration by parts. This shows that f is $\lambda \times \lambda$ integrable on E.

Proof (b) By Fubini's Theorem,

$$\int_E f(x, y)d(\lambda \times \lambda)(x, y) = \int_0^1 \int_0^\infty y \sin x\, e^{-xy} dxdy = \int_0^\infty \int_0^1 y \sin x\, e^{-xy} dydx.$$

Using integration by parts, one has

$$\int_0^\infty y \sin x\, e^{-xy} dx = \frac{y}{y^2 + 1}.$$

Hence,

$$\int_E f(x, y)d(\lambda \times \lambda)(x, y) = \int_0^1 \frac{y}{y^2 + 1} dy = \frac{1}{2}\log 2.$$

On the other hand, again by integration by parts one has,

$$\int_0^1 y \sin x\, e^{-xy} dy = \frac{\sin x}{x}\left(\frac{1 - e^{-x}}{x} - e^{-x}\right).$$

Therefore,

$$\int_0^\infty \frac{\sin x}{x}\left(\frac{1 - e^{-x}}{x} - e^{-x}\right) dx = \frac{1}{2}\log 2.$$