Measure and Integration: Solutions Final 2013-14

(1) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ Lebesgue measure. Determine the value of

$$\lim_{n\to\infty} \int_{(0,n)} (1+\frac{x}{n})^{-n}(1-\sin\frac{x}{n})\, d\lambda(x).$$

(2 pts)

Solution: Let $u_n(x) = \mathbf{1}_{(0,n)}(1+\frac{x}{n})^{-n}(1-\sin\frac{x}{n})$. The positive sequence $\big((1+\frac{x}{n})^{-n}\big)_n$ decreases to $e^{-x}\mathbf{1}_{(0,\infty)}$ and the sequence $\big(1-\sin\frac{x}{n}\big)_n$ is bounded from below by 0 and from above by 2 and converges to 1 as $n\to\infty$. Thus, $\lim_{n\to\infty} u_n(x) = \mathbf{1}_{(0,\infty)}e^{-x}$, and $0 \le u_n(x) \le 2(1+\frac{x}{2})^{-2}\mathbf{1}_{(0,\infty)}$ for $n \ge 2$ and all $x \in \mathbb{R}$. Since the function $2(1+\frac{x}{2})^{-2}\mathbf{1}_{(0,\infty)}$ is measurable, non-negative and the improper Riemann integrable on $(0,\infty)$ exists, it follows that it is Lebesgue integrable on $(0,\infty)$. By Lebesgue Dominated Convergence Theorem (and taking the limit for $n \ge 2$), we have

$$\begin{aligned}\lim_{n\to\infty} \int_{(0,n)} (1+\frac{x}{n})^{-n}(1-\sin\frac{x}{n})\, d\lambda(x) &= \lim_{n\to\infty} \int u_n(x) d\lambda(x) \\ &= \int \mathbf{1}_{(0,\infty)}e^{-x}\, d\lambda(x) = \int_0^\infty e^{-x}\, dx = 1.\end{aligned}$$

(2) Let $(X, \mathcal{F}, \mu)$ be a **finite** measure space. Assume $f \in \mathcal{L}^2(\mu)$ satisfies $0 < ||f||_2 < \infty$, and let $A = \{x \in X : f(x) \neq 0\}$. Show that

$$\mu(A) \ge \frac{(\int f\, d\mu)^2}{\int f^2\, d\mu}.$$

(1.5 pts)

Solution: Since $f = 0$ on A^c, we have $\int f\, d\mu = \int f\mathbf{1}_A\, d\mu$. Since μ is a finite measure and $(\mathbf{1}_A)^2 = \mathbf{1}_A$, then

$$||\mathbf{1}_A||_2 = (\mu(A))^{1/2} < \infty.$$

Thus, $\mathbf{1}_A \in \mathcal{L}^2(\mu)$ and by Hölder's inequality

$$\int f\, d\mu \le ||f||_2||\mathbf{1}_A||_2 = |f||_2(\mu(A))^{1/2}.$$

Squaring both sides and dividing by

$$||f||_2^2 = \int f^2\, d\mu\, (> 0),$$

we get

$$\mu(A) \ge \frac{(\int f\, d\mu)^2}{\int f^2\, d\mu}.$$

(3) Let $E = \{(x,y) : y < x < 1,\ , 0 < y < 1\}$. We consider on E the restriction of the product Borel σ-algebra, and the restriction of the product Lebesgue measure $\lambda \times \lambda$. Let $f : E \to \mathbb{R}$ be given by $f(x,y) = x^{-3/2}\, \cos(\frac{\pi y}{2x})$.

(a) Show that f is $\lambda \times \lambda$ integrable on E. (0.5 pt)

(b) Define $F : (0,1) \to \mathbb{R}$ by $F(y) = \int_{(y,1)} x^{-3/2}\, \cos(\frac{\pi y}{2x})\, d\lambda(x)$. Determine the value of

$$\int F(y)\, d\lambda(y).$$

(2 pts)

Solution (a) : Notice that f is continuous, and hence measurable. Furthermore, $|f(x,y)| \leq x^{-3/2}$. The function $g(x,y) = x^{-3/2}$ is non-negative and measurable on E, hence by Tonelli's Theorem,

$$\begin{aligned}\int_E |f(x,y)|\, d(\lambda\times\lambda)(x,y) &\leq \int_E g(x,y)\, d(\lambda\times\lambda)(x,y)\\ &= \int_0^1\int_0^x x^{-3/2}\, dy\, dx\\ &= \int_0^1 x^{-1/2}\, dx = 2.\end{aligned}$$

Notice that the integrands are Riemann integrable, hence the Riemann integral equals the Lebesgue integral. This shows that f is $\lambda\times\lambda$ integrable on E.

Solution (b) : By Fubini's Theorem

$$\int\int f(x,y)\, d\lambda(x)\, d\lambda(y) = \int\int f(x,y)\, d\lambda(y)\, d\lambda(x).$$

Notice that for each fixed $0 < x < 1$, the function $f(x,y)$ is Riemann-integrable in y on the interval $(0,x)$ and

$$\int_0^x x^{-3/2}\cos(\frac{\pi y}{2x})\, dy = \frac{2}{\pi}\, x^{-1/2},$$

and the function $\frac{2}{\pi}x^{-1/2}$ is Riemann-integrable in x on the interval $(0,1)$, and

$$\int_0^1 \frac{2}{\pi}\, x^{-1/2}\, dx = \frac{4}{\pi}.$$

Thus,

$$\int F(y)\, d\lambda(y) = \int\int f(x,y)\, d\lambda(x)\, d\lambda(y) = \int_0^1\int_0^x x^{-3/2}\cos(\frac{\pi y}{2x})\, dy\, dx = \frac{4}{\pi}.$$

(4) Let $1 \leq p < \infty$, and suppose $(X,\mathcal{A},\mu)$ is a measure space. Let $(f_n)_n \in \mathcal{L}^p(\mu)$ be a sequence converging to f in $\mathcal{L}^p$ i.e. $\lim_{n\to\infty} ||f_n - f||_p = 0$.

(a) Show that

$$\int |f|^p\, d\mu \leq \liminf_{n\to\infty}\int |f_n|^p\, d\mu.$$

(1 pt)

(b) Show that $\lim_{n\to\infty} n^p\mu(\{|f| > n\}) = 0$. (1 pt)

Solution (a): This is a simple consquence of the triangle inequality applied to the $\mathcal{L}_p$-norm and in fact the lim inf can be replaced by lim and the inequality by equality, namely

$$|||f_n||_p - ||f||_p| \leq ||f_n - f||_p.$$

Taking limits, we get the desired result. (Remark: if we replace $\mathcal{L}_p$-convergence by convergence in measure, then the inequality is really needed).

Solution (b): Note that $f \in \mathcal{L}^p(\mu)$ and hence by Corollary 10.13,

$$\mu(\{|f|^p = \infty\}) = \mu(\{|f| = \infty\}) = 0.$$

Thus,

$$\lim_{n\to\infty} |f|^p\mathbf{1}_{\{|f|>n\}} = |f|^p\mathbf{1}_{\{|f|=\infty\}} = 0\ \ \mu\ a.e.$$

Since for each n, $|f|^p\mathbf{1}_{\{|f|>n\}} \leq |f|^p$ and $|f|^p \in \mathcal{L}^1(\mu)$, we have by Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int |f|^p\mathbf{1}_{\{|f|>n\}}\, d\mu = 0.$$

Now,

$$n^p\mu(\{|f| > n\}) = \int n^p\mathbf{1}_{\{|f|>n\}}\, d\mu \leq \int |f|^p\mathbf{1}_{\{|f|>n\}}\, d\mu,$$

and from the above we get $\lim_{n\to\infty} n^p\mu(\{|f| > n\}) = 0$.

(5) Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $f_n, f \in \mathcal{M}(\mathcal{A})$, $n \geq 1$. Show that f_n converges to f in μ measure **if and only if** $\lim_{n\to\infty} \int \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu = 0$. (2 pts)

Solution: First note that $\frac{|f_n - f|}{1 + |f_n - f|} \leq 1$ for all $n \geq 1$, and since $\mu(X) < \infty$ we have $1 \in \mathcal{L}^1(\mu)$.

Now assume that $f_n \xrightarrow{\mu} f$, and let $\epsilon, \delta > 0$, then there exists N such that

$$\mu(\{x \in X : |f_n(x) - f(x)| > \delta\}) < \epsilon, \quad \text{for all } n \geq N.$$

Let $A = \{x \in X : |f_n(x) - f(x)| > \delta\}$, then for all $n \geq N$

$$\int \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu = \int_A \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu + \int_{A^c} \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu \leq \int_A 1\, d\mu + \int_{A^c} \delta\, d\mu.$$

Thus, for all $n \geq N$

$$\int \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu \leq \epsilon + \delta\mu(X).$$

Thus, $\lim_{n\to\infty} \int \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu = 0$.

Conversely, assume $\lim_{n\to\infty} \int \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu = 0$, and let $\epsilon > 0$. There exists N such that

$$\int \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu < \epsilon^2/(1 + \epsilon), \quad \text{for all } n \geq N.$$

Observe first that

$$|f_n - f| > \epsilon \iff \frac{|f_n - f|}{1 + |f_n - f|} > \frac{\epsilon}{1 + \epsilon}.$$

Thus, by Markov Inequality, we have for all $n \geq N$

$$\mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) = \mu(\{x \in X : \frac{|f_n - f|}{1 + |f_n - f|} > \frac{\epsilon}{1 + \epsilon}\}) \leq \frac{1 + \epsilon}{\epsilon} \int \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu < \epsilon.$$

Thus, $f_n \xrightarrow{\mu} f$.