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Measure and Integration: Final 2015-16

(1) Consider the measure space $[0,1],\mathcal{B}([0,1]),\lambda)$ where λ is Lebesgue measure on $[0,1]$. Define $u_n(x)=\dfrac{n^2x^2}{1+n^2x}$ for $x\in[0,1]$ and $n\geq 1$. Show that

$$\lim_{n\to\infty}\int_{[0,1]}\frac{n^2x^2}{1+n^2x}\,d\lambda(x)=1/2.$$

(1 pt)

(2) Suppose μ and ν are finite measures on $(X,\mathcal{A})$. Show that there exists a function $f\in\mathcal{L}^1_+(\mu)$, and a set $A_0\in\mathcal{A}$ with $\mu(A_0)=0$ such that

$$\nu(E)=\int_E f\,d\mu+\nu(A_0\cap E),$$

for all $E\in\mathcal{A}$. (1.5 pts)

(3) Consider the measure space $[0,1),\mathcal{B}([0,1)),\lambda)$ where λ is Lebesgue measure on $[0,1)$. Let $D_1=[0,1/2)$ and $D_k=\left[\sum_{i=1}^{k-1}2^{-i},\sum_{i=1}^{k}2^{-i}\right)$, $k\geq 2$. Define $u(x)=\sqrt{2^{k-1}}$ for $x\in D_k$, $k\geq 1$. Determine the values of $p\in[1,\infty)$ such that $u\in\mathcal{L}^p(\lambda)$. In case $u\in\mathcal{L}^p(\lambda)$, find the value of $||u||_p$. (2 pts.)

(4) Let $(X,\mathcal{A},\mu)$ be a σ-finite measure space, and Let $(u_j)_j\subseteq\mathcal{L}^1(\mu)$. Suppose $(u_j)_j$ converges to u μ a.e., and that the sequence (u_j^-) is uniformly integrable. Prove that $\liminf_{n\to\infty}\int u_n\,d\mu\geq\int u\,d\mu$. (2 pts)

(5) Let $(X,\mathcal{A},\mu)$ be a σ-finite measure space, and assume $u\in\mathcal{M}^+(\mathcal{A})$. Let $\phi:[0,\infty)\to\mathbb{R}$ be continuously differentiable (i.e. ϕ' exists and is continuous) such that $\phi(0)=0$ and $\phi'\geq 0$ for all $t\geq 0$. Show that

$$\int_X\phi\circ u(x)\,d\mu=\int_{[0,\infty)}\phi'(t)\mu(\{x\in X:u(x)\geq t\})\,d\lambda(t).$$

Conclude that if $u\in\mathcal{L}^p_+(\mu)$, then

$$\int_X u^p\,d\mu=p\int_{[0,\infty)}t^{p-1}\mu(\{x\in X:u(x)\geq t\})\,d\lambda(t).$$

(2 pts)

(6) Let $(X,\mathcal{A},\mu)$ be a measure space and $f\in\mathcal{L}^1(\mu)\cap\mathcal{L}^2(\mu)$.
   (a) Show that $f\in\mathcal{L}^p(\mu)$ for all $1\leq p\leq 2$. (1 pt)
   (b) Prove that $\lim_{p\searrow 1}||f||_p^p=||f||_1$. (1 pt)